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Study of Three Dimensional Quantum Black Holes*

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Abstract

We investigate quantum aspects of the three dimensional black holes discovered by Bañados, Teitelboim and Zanelli. The discussions are devoted to two subjects: the thermodynamics of quantum scalar fields and the string theory in the three dimensional black hole backgrounds. We take two approaches to the thermodynamics. In one approach we use mode expansion, and in the other we use Hartle-Hawking Green functions. We obtain exact expressions of mode functions, Hartle-Hawking Green functions, Green functions on a cone geometry, and thermodynamic quantities. The entropies depend largely upon methods of calculation including regularization schemes and boundary conditions. This indicates the importance of precise discussions on the definition of the thermodynamic quantities for understanding black hole entropy. Then we investigate the string theory in the framework of conformal field theory. The model is described by an orbifold of the $\widehat{SL}(2, R)$ WZW model. We discuss the spectrum by solving the level matching condition and obtain winding modes. We analyze the physical states and investigate the ghost problem. Explicit examples of negative-norm physical states (ghosts) are found. Thus we discuss possibilities for obtaining a sensible theory. The tachyon propagation and the target-space geometry are also discussed. This is the first attempt to quantize a string in a black hole background with an infinite number of propagating modes. Although we cannot overcome all the problems, our results may provide a useful basis for both subjects.

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1 INTRODUCTION

1.1 Black holes and quantum gravity

In the early 1960s, a rapid development began in general theory of relativity [1]. New mathematical techniques simplified calculation of physical quantities. The progress of technology enabled us to carry out various experiments. They changed general relativity into a tractable physics from a profound but abstract theory. By the end of the 1970s, we had obtained a deeper understanding of general relativity [1]-[3]. This theory passed every experimental test. New astronomical objects (quasars, pulsars, black holes) and the cosmic background radiation were discovered. We found evidence of the existence of gravitational waves from a binary pulsar. Furthermore, global structure of space-time and properties of singularities and black holes were clarified. A number of exact solutions were also found.

In parallel with this progress, another excellent development took place in elementary particle physics. After a skeptic period over quantum field theory, the standard model was established. Precise experiments became possible, and they revealed the nature of physics at very small scale. The experiments stimulated the progress of the theory, and vice versa. At last, based on simple principles, i.e., Lorentz invariance, the gauge principle and renormalizability, we found a beautiful theory of elementary particles. We obtained the basic law which describes fundamental interactions except for gravity in a quantum mechanical manner.

Through these developments, we acquired a profound insight into nature. Natural questions then arise: Can we go beyond the standard model ? Can we construct the theory of quantum gravity ? These are important and challenging subjects in physics today.

In the investigation of quantum gravity, black holes are regarded as an excellent arena. They include a strong coupling region of gravity where quantum effects may become important. We can easily draw a physical picture of them. Moreover, close relations between black hole physics and quantum theory have been found through the study of black hole thermodynamics.

Black holes obey analogs of the laws of thermodynamics [3]-[5], and this is called black hole thermodynamics. At first sight, the analogy seems superficial since we have no reason that black holes are thermal. However, we find arguments for this analogy by semiclassical analyses. Hawking found that black holes emit thermal radiation by effects of quantum fields (Hawking radiation) [6]. Bekenstein argued that the second law of the black hole thermodynamics is valid for a system of gravity and quantum matter (generalized second law) [5, 7]. Thermal properties of black holes were further

discussed using the path integral and Green functions [8]-[10]. These imply connections among black hole physics, quantum mechanics and statistical mechanics. In addition, the Hawking radiation indicates a contradiction to the unitary evolution of quantum mechanics (the information paradox). Therefore, many works have been devoted to black hole thermodynamics for understanding it from the microscopic, quantum mechanical and statistical mechanical point of view. We expect to get clues to quantum gravity by the investigation of black holes. We also expect to understand fundamental problems concerning gravity, e.g., the problem of singularities and of the early universe.

Since string theory is the candidate of the fundamental theory including gravity, it should give the answer to the problems about quantum black holes. To this end, we have to develop analysis beyond low energy effective theory and the α' (Regge slope) expansion. The $SL(2, R)/U(1)$ black hole [11, 12] serves as a useful model in this respect. It gives an exact background of a string, and it is described by a simple Wess-Zumino-Witten (WZW) model [13]. Thus the properties of the $SL(2, R)/U(1)$ black hole have been studied extensively (see, e.g., [11]-[21]). Nevertheless we need further investigations to clarify important issues of black hole physics. The difficulties are rooted in the fact that the target space is non-compact and curved in time direction. Such difficulties are not characteristic of string theories in black hole backgrounds. They are typical of a string theory in a non-trivial background with curved time. Although we have many consistent string theories on curved spaces, i.e., on group manifolds, they are compact and must be tensored with Minkowski spacetime. There have been a few previous attempts besides the $SL(2, R)/U(1)$ case. For example, there are attempts using the $SL(2, R)$ WZW model [22]-[28], but it is known to contain ghosts.¹ So far, we have few consistent string theories with curved time.²

In this thesis, we investigate quantum aspects of the three dimensional black holes discovered by Bañados, Teitelboim and Zanelli (BTZ) [31]. The BTZ black hole is a solution to the vacuum Einstein field equations with a negative cosmological constant. It shares many characteristics with the $(3+1)$ -dimensional Kerr black hole (for a review, see, e.g., [32]). Moreover, it provides a very simple system: it has a constant curvature and no curvature singularities. Therefore we can discuss many characteristics of the black hole physics in an explicit manner without mathematical complications. Another importance of the BTZ black hole is the fact that this is one of the few known exact solutions in string theory and one of the simplest solutions [33, 34]. In addition, a string in three dimensions has an infinite number of propagating modes, so it resembles a higher dimensional one. This theory has significance also as a string theory in non-trivial curved spacetime. Thus we may obtain useful insights into important issues of quantum gravity through the study

¹A resolution to the ghost problem has recently been proposed in [29].

²See, however, [30] for instance.

of the string theory in the BTZ black hole background.

The aim of this thesis is to investigate quantum gravity by the analysis of quantum aspects of the three dimensional black hole. In particular, we have two main purposes in this thesis. One is to investigate the thermodynamics of quantum fields in the three dimensional black hole background [35]. Recently, black hole thermodynamics has attracted attention again [36]-[40]. These works include suggestive arguments for the microscopic origin of black hole entropy and for the relation among the information paradox, the black hole entropy and the renormalization of the gravitational coupling constant. However, the arguments seem formal and deal mainly with the flat-space limit. In our case, we can study this issue in a truly curved background and in an explicit manner. We get exact expressions and discuss the problems without ambiguity. We find that thermodynamic quantities depend largely upon methods of calculation and that the results concerning divergences and the role of horizons do not necessarily agree with [36]-[40]. These indicate the importance of curvature effects and precise discussions on the definition of the thermodynamic quantities including regularization schemes and boundary conditions. We need further investigations in these respects. Our results, however, serve as a reliable basis for the quantum field theory and the thermodynamics of quantum scalar fields in the BTZ black hole background.

The other purpose is to investigate the string theory in the three dimensional black hole geometry [41]. One of the motivations is to settle the open problems of the thermodynamics of the quantum fields and to clarify the microscopic origin of black hole thermodynamics. However, the purpose here is more general as explained above. In spite of its importance, detailed analyses have not been made so far. This is partly because we do not know much about string theory in curved spacetime and it is not clear whether the string theory in the BTZ background satisfies consistency conditions as a sensible theory. We investigate this string theory in detail in the framework of conformal field theory. We analyze the spectrum and obtain winding modes. We study the physical states. We examine the ghost problem and find explicit examples of ghosts. This means that our model is not unitary as it stands. Thus we discuss possibilities for obtaining a sensible theory. The tachyon propagation and the target-space geometry are also discussed. Although we cannot overcome all the problems, our work may provide a starting point for further investigations of this issue.

1.2 Organization of the thesis

This thesis is organized as follows.

In chapter 2, we review the three dimensional (BTZ) black holes.

In chapter 3, we discuss the thermodynamics of scalar fields in the BTZ black hole

background. In section 3.1, we briefly review black hole thermodynamics and recent arguments about black hole entropy which are relevant to our discussion. We take two approaches to the thermodynamics of the scalar field. In section 3.2, we utilize mode expansion and summation over states. We find exact mode functions and thermodynamic quantities. In section 3.3, we construct Hartle-Hawking Green functions. We investigate the thermodynamics based on the Euclidean Hartle-Hawking Green functions in section 3.4. We obtain free energies and Green functions on a cone geometry. By using them, we calculate entropies. These are also exact in the framework of quantum field theory in curve spacetime. Discussion on our results is given in section 3.5.

In chapter 4, we investigate the string theory in the three dimensional black hole geometry. In section 4.1, we review the BTZ black hole from the string-theory point of view. In section 4.2, we construct the orbifold of the $\widetilde{SL}(2, R)$ WZW model which describes the string in the black hole background. We analyze the spectrum by solving the level matching condition. We obtain winding modes. In section 4.3, we investigate the ghost problem. We find explicit examples of negative-norm physical states. In section 4.4, we discuss the tachyon propagation and the target-space geometry. We find a T-duality transformation reversing the black hole mass. In section 4.5, we discuss possibilities for obtaining a sensible theory.

We conclude this thesis in chapter 5.

Technical details and mathematical backgrounds are collected in appendix A-D. In appendix A, we summarize the derivation of the Feynman Green function in \widetilde{AdS}_3 . In appendix B, we derive Green functions on a cone geometry by using the Sommerfeld integral representation. In appendix C, we collect basic properties of the representation theory of $\widetilde{SL}(2, R)$. Representations in the hyperbolic basis are explained in some detail. Finally in appendix D, we show the Clebsch-Gordan decomposition of the $sl(2, R)$ Kac-Moody module in the hyperbolic basis.

2 THE THREE DIMENSIONAL BLACK HOLE

In this section, we review the three dimensional black hole discovered by Bañados et al. (the BTZ black hole) [31].

We begin with the three dimensional anti-de Sitter space (AdS_3). AdS_3 is a three dimensional hyperboloid embedded in a flat space with the metric

$$ds^2 = -dx_0^2 - dx_1^2 + dx_2^2 + dx_3^2, \quad (2.1)$$

through the equation

$$-x_0^2 - x_1^2 + x_2^2 + x_3^2 = -l^2. \quad (2.2)$$

It is a maximally symmetric space and forms a solution to Einstein gravity with a negative cosmological constant $-l^{-2}$. The curvature tensors are

$$R_{\mu\nu} = -2l^{-2}g_{\mu\nu}, \quad R = -6l^{-2}.$$

Notice that the scalar curvature is constant. In order to decompactify the time direction of AdS_3 , we go to the universal covering space \widetilde{AdS}_3 . We then consider three regions parametrized by

$$\begin{aligned} \text{Region I} \quad (\hat{r}^2 > l^2) \quad &: x_1 = \hat{r} \cosh \hat{\varphi}, & x_0 = \sqrt{\hat{r}^2 - l^2} \sinh \hat{t}, \\ & x_2 = \hat{r} \sinh \hat{\varphi}, & x_3 = \sqrt{\hat{r}^2 - l^2} \cosh \hat{t}, \\ \text{Region II} \quad (l^2 > \hat{r}^2 > 0) \quad &: x_1 = \hat{r} \cosh \hat{\varphi}, & x_0 = \sqrt{l^2 - \hat{r}^2} \cosh \hat{t}, \\ & x_2 = \hat{r} \sinh \hat{\varphi}, & x_3 = \sqrt{l^2 - \hat{r}^2} \sinh \hat{t}, \\ \text{Region III} \quad (0 > \hat{r}^2) \quad &: x_1 = \sqrt{-\hat{r}^2} \sinh \hat{\varphi}, & x_0 = \sqrt{l^2 - \hat{r}^2} \cosh \hat{t}, \\ & x_2 = \sqrt{-\hat{r}^2} \cosh \hat{\varphi}, & x_3 = \sqrt{l^2 - \hat{r}^2} \sinh \hat{t}, \end{aligned} \quad (2.3)$$

where $-\infty < \hat{t}, \hat{\varphi} < \infty$. In every parametrization, substituting it into (2.1) yields the metric of \widetilde{AdS}_3 of the form

$$ds^2 = -\left(\frac{\hat{r}^2}{l^2} - 1\right)d\hat{t}^2 + \hat{r}^2 d\hat{\varphi}^2 + \left(\frac{\hat{r}^2}{l^2} - 1\right)^{-1}d\hat{r}^2.$$

$\partial_{\hat{t}}$ and $\partial_{\hat{\varphi}}$ generate isometries. These correspond to boost symmetries in the flat space. We make a further change of variables,

$$\frac{\hat{r}^2}{l^2} = \frac{r^2 - r_-^2}{d_H^2}, \quad \begin{pmatrix} \hat{t} \\ \hat{\varphi} \end{pmatrix} = \frac{1}{l} \begin{pmatrix} r_+ & -r_- \\ -r_- & r_+ \end{pmatrix} \begin{pmatrix} t/l \\ \varphi \end{pmatrix}, \quad (2.4)$$

where r_{\pm} ($r_+ > r_- \geq 0$) are positive constants, and

$$d_H^2 = r_+^2 - r_-^2.$$

Then, by identifying the points under a discrete subgroup of an isometry

$$\varphi \sim \varphi + 2\pi n \quad (n \in \mathbf{Z}),$$

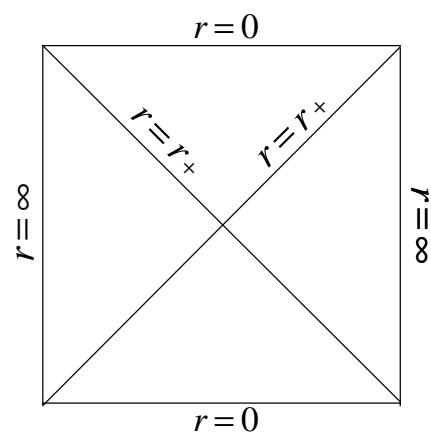
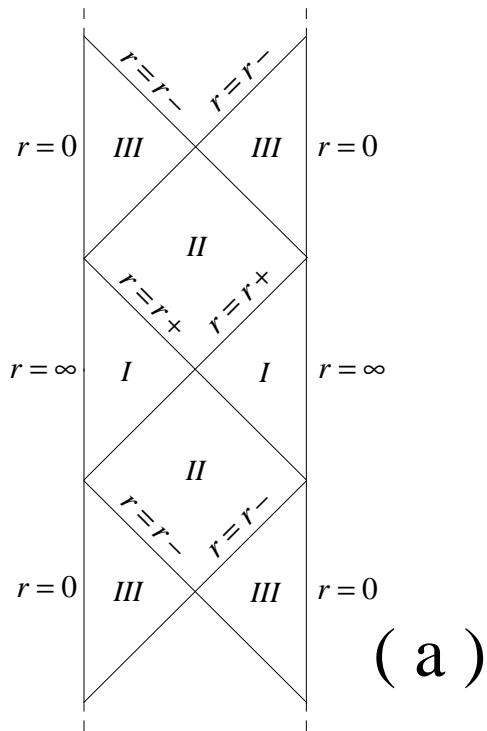
one obtains the geometry of the three dimensional black hole:

$$\begin{aligned} ds_{BH}^2 &= g_{\mu\nu}^{BH} dx^{\mu} dx^{\nu} \\ &= -N_{\perp}^2 dt^2 + N_{\perp}^{-2} dr^2 + r^2(N^{\varphi} dt + d\varphi)^2, \\ &= -\left(\frac{r^2}{l^2} - M_{BH}\right)dt^2 - J_{BH} dt d\varphi + r^2 d\varphi^2 + \left(\frac{r^2}{l^2} - M_{BH} + \frac{J_{BH}^2}{4r^2}\right)^{-1}dr^2, \end{aligned} \quad (2.5)$$

where

$$\begin{aligned} N_{\perp}^2 &= \frac{(r^2 - r_+^2)(r^2 - r_-^2)}{l^2 r^2}, \quad N^{\varphi} = -\frac{r_+ r_-}{l r^2}, \\ l^2 M_{BH} &= (r_+^2 + r_-^2), \quad l J_{BH} = 2r_+ r_-. \end{aligned}$$

The coordinates in (2.5) now take $-\infty < t < +\infty$, $0 \leq \varphi < 2\pi$ and $0 \leq r < +\infty$. Under the above identification, $r^2 = 0$ is the fixed point for $J_{BH} = 0$, but for $J_{BH} \neq 0$ one has no fixed points.



(b)

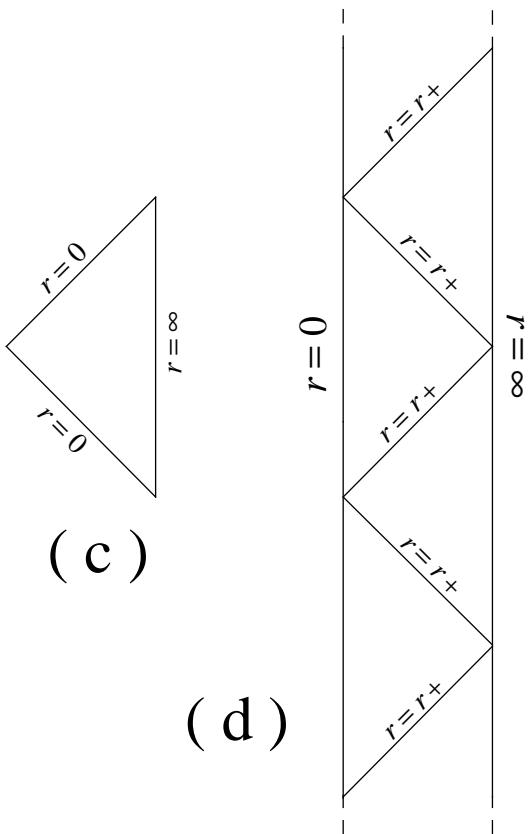


Figure 1a - 1d :

Conformal diagrams for

- (a) $J_{BH} \neq 0$ and $J_{BH}/l \neq M_{BH}$,
- (b) $J_{BH} = 0$ and $M_{BH} \neq 0$,
- (c) $M_{BH} = J_{BH}/l = 0$,
- (d) $M_{BH} = J_{BH}/l \neq 0$.

The above metric describes a non-extremal rotating black hole for $r_- \neq 0, r_+$. M_{BH} and J_{BH} represent the mass and the angular momentum of the black hole, respectively. The black hole has two horizons given by the surfaces $r = r_\pm$. A conformal diagram for $r_- \neq 0, r_+$ is shown in Figure 1a. For $r_- = 0$, the angular momentum J_{BH} vanishes and the black hole becomes non-rotating one. A conformal diagram for this case is given in Figure 1b. In the metric (2.5), we can take the limit $r_- \rightarrow r_+$ ($J_{BH}/l \rightarrow M_{BH}$) although singular quantities appear in the intermediate expressions. The resulting geometry describes an extremal black hole (Figure 1c and 1d). The way to obtain the extremal black hole by an identification of \widetilde{AdS}_3 is quite different from the other cases. In this thesis, we will focus on the non-extremal cases.

In the conformal diagrams, we have cut out the region $r^2 < 0$ where closed timelike curves exist. There are arguments (i) that the inclusion or non-inclusion of this region is irrelevant to an observer outside the black hole because the surface $r = r_+$ is the event horizon, and (ii) that the inclusion of matter produces a curvature singularity at $r = 0$ (or $r = r_-$) and one *has to* drop that region [31, 42, 43]. We will briefly discuss this issue later in the context of string theory.

Since the BTZ black hole is locally \widetilde{AdS}_3 , it is also a solution to Einstein gravity. The asymptotic region tends to be \widetilde{AdS}_3 instead of Minkowski spacetime. The curvature is constant and there are no curvature singularities; namely, the BTZ black hole provides a very simple system. Thus, through its analysis, we can investigate many characteristics of black hole physics in an explicit manner without mathematical complications (for a review, see [32]). Indeed, its properties have been studied extensively in the classical theory. For example, the BTZ black hole shares many characteristics with the $(3+1)$ -dimensional Kerr black hole: it has an event horizon, an inner horizon and an ergosphere; it occurs as an end point of “gravitational collapse”; it exhibits instability of the inner horizon; and it has a non-vanishing Hawking temperature and various thermodynamic properties. The thermodynamic quantities are given by [31]

$$\beta_H = \frac{2\pi r_+ l^2}{r_+^2 - r_-^2}, \quad S_{BH} = \frac{4\pi r_+}{l}, \quad \nu_{BH} = \frac{r_-}{r_+ l}, \quad (2.6)$$

where β_H is the inverse temperature; S_{BH} is the entropy; and ν_{BH} is the chemical potential conjugate to J_{BH} . Moreover, the utility of the BTZ black hole becomes evident in the quantum analysis. Quantum field theory in the BTZ black hole background has been explored and exact results have been obtained about Green functions for a conformally coupled massless scalar. The thermodynamic and statistical mechanical properties of the black hole have been investigated by the Chern-Simons formulation of the $(2+1)$ -dimensional general relativity.

The BTZ black hole is also one of the few known exact solutions in string theory and one of the simplest ones. Since a string in a BTZ black hole background has an infinite

number of propagating modes, the string theory resembles a higher dimensional one. It has significance also as a string theory on a non-trivial curved spacetime.

In the rest of this thesis, we will investigate the thermodynamics of quantum scalar fields [35] and the string theory [41] in the BTZ black hole background.

3 THERMODYNAMICS OF SCALAR FIELDS IN THE THREE DIMENSIONAL BLACK HOLE BACKGROUND

In this section, we discuss the thermodynamics of quantum scalar fields in the BTZ black hole background [35]. After a brief review of black hole thermodynamics,³ we discuss the thermodynamics of the scalar field in two approaches. One is to use mode expansion, and the other is to use Hartle-Hawking Green functions. In both approaches we obtain exact expressions. These enable us to discuss the black hole thermodynamics without ambiguity.

3.1 Black hole entropy

Through the study of black hole thermodynamics, close relations between black holes and quantum mechanics have been found [3]-[10]. In addition, black hole entropy has lately attracted attention again [36]-[40]. In order to make the points of later discussions clear, we first review black hole thermodynamics and recent arguments about the entropy of a quantum field in a black hole background.

3.1.1 Black hole thermodynamics

In general relativity, black holes satisfy the following properties under certain conditions [3]-[5]:

0. κ (surface gravity) is constant over the horizon of a black hole.
1. In physical processes, the changes of physical quantities obey

$$\delta M = (\kappa/8\pi)\delta A + \Omega_H\delta J + \Phi_H\delta Q,$$

where A , J and Q are the area, the angular momentum and the charge of the black hole, respectively. Ω_H is the angular velocity and Φ_H is the potential at the horizon.

2. The area of the horizon never decreases, $\delta A \geq 0$.

³For a review of black hole thermodynamics and recent arguments about black hole entropy in canonical quantum gravity, see, e.g.,[3, 44, 45].

3. It is impossible to achieve $\kappa = 0$.

These remind us of the ordinary laws of thermodynamics. The correspondences are $M \leftrightarrow E$ (energy), $\alpha\kappa \leftrightarrow T$ (temperature) and $A/8\pi \leftrightarrow S$ (entropy), where α is some constant. A gravitationally collapsing body rapidly settles to a black hole configuration specified only by three macroscopic quantities M , J and Q . This is also suggestive of this analogy. But, the temperature of a black hole is classically zero, and the relationship between the area and the entropy is obscure. Thus the analogy seems superficial at first. Nevertheless, if one makes semiclassical analyses including quantum fields in black hole backgrounds, one finds evidence that black holes are really thermal. Let us see the representative arguments.

Hawking radiation [6]

By evaluating the vacuum expectation value of the number operator of a quantum field, Hawking showed that a black hole emits thermal radiation at temperature $T = \kappa/2\pi$. This argument fixes the constant α to be $1/2\pi$. The entropy of the black hole is then given by

$$S_{BH} = \frac{1}{4l_p^2}A = \frac{1}{4G}A = \frac{1}{4}A,$$

where l_p is the Planck length and G is the gravitational coupling constant. We have displayed the formula in various units. (This is called Bekenstein-Hawking entropy.)

The Hartle-Hawking Green function [8]

One can define a Green function of a quantum field in a black hole geometry by a generalization of the Feynman Green function in Minkowski spacetime. This is specified by the analyticity or the boundary condition at the horizon. This is called the Hartle-Hawking Green function. By making use of this Green function, one can derive the Hawking radiation again.

Euclidean black holes [9]

By Wick-rotating the Schwarzschild metric by $\tau \equiv it$, one obtains a “Euclidean” black hole geometry

$$ds_E^2 = R^2 d(\tau/4M)^2 + (r/2M)^4 dR^2 + r^2 d\Omega^2,$$

where $R \equiv 4M(1 - 2M/r)^{1/2}$. τ then has the period $\beta_H \equiv 8\pi M$. Suppose that we define a Green function in the original Schwarzschild geometry by an analytic continuation (as in an ordinary field theory); $G(x, y) \equiv G^E(x^E, y^E) \Big|_{\tau=it}$. Then $G(x, y)$ has an imaginary period $t \rightarrow t + i\beta_H$. (This is nothing but the Hartle-Hawking Green function.) An imaginary period of the time direction is a characteristic feature of thermal Green functions.

Thus the above Green function suggests that the quantum field is in thermal equilibrium at temperature $T = \beta_H^{-1} = \kappa/2\pi$ ($\kappa = 1/4M$).

Gibbons-Hawking entropy [10]

In Minkowski spacetime, the partition function of a quantum field Φ at inverse temperature β is given by

$$Z(\beta) = \text{Tr } e^{-\beta H} = \int \mathcal{D}\Phi \exp \left\{ - \int_0^\beta d\tau \int d\vec{x} \sqrt{g_E} \mathcal{L}^E(\Phi) \right\}. \quad (3.1)$$

Here H is the Hamiltonian and \mathcal{L}^E is the Euclidean Lagrangian density. The path integral is performed under the periodic boundary condition $\Phi(\tau) = \Phi(\tau + \beta)$. Suppose that this expression is valid for the gravitational field in the Schwarzschild geometry, and evaluate the path integral at $\beta = \beta_H$ in the saddle point approximation (the saddle point corresponds to the Euclidean Schwarzschild solution). Then, one obtains $S_{BH} = A/(4G)$ again. Since the action of the gravitational part is unbounded below, it is necessary to regularize it.

For a matter field, such a calculation corresponds to defining the partition function Z_m and the entropy S_m by

$$\begin{aligned} \ln Z_m(\beta) &= -\beta F_m(\beta) = \frac{1}{2} \text{Tr } \ln G_m^E(\beta), \\ S_m(\beta) &= \beta^2 \frac{\partial F_m}{\partial \beta}, \end{aligned} \quad (3.2)$$

where F_m is the free energy and G_m^E is the Euclidean Green function of the matter in the Schwarzschild background. Note that the Euclidean black hole geometry has a deficit angle with respect to τ if $\beta \neq \beta_H$. In this case, (τ, r) -plane represents a “cone geometry”.

Generalized second law [5, 7]

If the area of the horizon $(1/4)A$ can be interpreted as the thermodynamic entropy, the total entropy of gravity plus matter should not decrease:

$$\delta S^{tot} \equiv \delta S^m + (1/4)\delta A \geq 0.$$

This is called the generalized second law. Although the proof does not exist, there are arguments for the validity of this law.

These semiclassical arguments materialize black hole thermodynamics. Again by an analogy to the ordinary thermodynamics, the microscopic meaning of which is given by statistical mechanics, we expect the existence of some fundamental and microscopic mechanism of black hole thermodynamics. In addition, the Hawking radiation implies

that quantum coherence may not hold in quantum gravity. This is a serious problem for fundamental theory of physics. In this way, black hole thermodynamics indicates deep connection among gravity, quantum theory and statistical physics. Thus, this subject has been studied extensively to get clues to quantum gravity. In particular, main problems have been (i) to derive the black hole entropy by counting quantum states of black holes⁴ (ii) to explain the Hawking radiation in a quantum mechanical manner and (iii) to prove the generalized second law.

3.1.2 Recent arguments about black hole entropy

Let us turn to the recent arguments about black hole entropy. Since we have no consistent theory of quantum gravity, it is difficult to make definite arguments about black hole thermodynamics, in particular about the gravitational part. However, there are recent proposals to explain the black hole entropy in quantum mechanical manners for matter fields. In the following, we will list examples of these proposals [36]-[40] relevant to the later discussions.⁵ They deal with the limit where black hole mass goes to infinity. Note that the Schwarzschild geometry approaches Rindler space in that limit (Rindler limit).

The brick wall model [47],[37, 38]

Let us consider a quantum scalar field $f(x)$ with mass $m \ll 1$ in a Schwarzschild black hole geometry with mass $M \gg 1$. The horizon is at $r = 2M$. We impose the boundary condition $f(x) = 0$ at $r = 2M + \epsilon, L$, where $\epsilon \ll 1$ and $L \gg M$. It turns out that ϵ and L play the role of regulators of the thermodynamic quantities. This boundary condition is similar to putting a “brick wall” near the horizon. Then we calculate the density of states $dg(E)$ by the WKB approximation, and the free energy by summation over states as $\beta F(\beta) = \int_0^\infty dE (dg/dE) \ln(1 - e^{-\beta E})$. Consequently, one finds that (i) the entropy of the matter is proportional to the area of the horizon A , and (ii) the leading contribution to the entropy comes from the horizon and diverges as A/ϵ^2 . This entropy is regarded as a quantum correction to the Bekenstein-Hawking entropy ($A/(4l_p^2)$). These results are suggestive of the origin of black hole entropy, and they imply the importance of the horizon to quantum properties of black holes. However, it is not clear why we should adopt such a boundary condition.

Geometric entropy [48],[36, 37, 39]

A black hole has an event horizon, and this separates the outside and the inside region; an observer outside the horizon cannot get any information from the inside. One can infer

⁴In this respect, interesting results have been obtained recently for black holes in super string theory [46].

⁵There are many other recent arguments. We do not refer to them here.

that this can be a source of the thermodynamic properties because the inside information may be averaged for the outside observer. This is the essential idea of geometric entropy.

Let us consider a free scalar field in a *flat* spacetime separated into two parts by a boundary with thickness ϵ . We calculate the density matrix (by introducing appropriate cutoffs), and average the states inside the boundary by taking the trace of the density matrix. Then one finds that (i) the entropy calculated from the averaged density matrix is proportional to the area of the boundary A , and (ii) the leading contribution to the entropy comes from the boundary and diverges as A/ϵ^2 . These results are suggestive again, and they imply the importance of the correlation between the outside and the inside. Notice that gravitation is absent in this calculation. Thus the relation to the real black hole physics is not clear although this argument is relevant in the Rindler (flat) limit.

Gibbons-Hawking entropy of matter [38]-[40]

For a free scalar field in a Euclidean black hole geometry, one can evaluate the path integral in (3.1) systematically by the heat kernel expansion. We remark that one has to consider the heat kernel on a cone geometry for $\beta \neq \beta_H$, and hence for getting the entropy (see (3.2)). In the Rindler limit, one then finds that (i) the entropy is proportional to the area of the horizon A , and (ii) this is divergent like A/ϵ^2 where ϵ is an ultra-violet cutoff. This entropy is regarded as a quantum correction to the Bekenstein-Hawking entropy again. From this point of view, one can regard the entropies as responses to the deficit angle of a geometry.

At first sight, the relation among these calculations is not obvious. But one can show the equivalence of these three methods in the Rindler limit [37, 39]. This is reasonable because the three methods give the same result in Minkowski spacetime and Rindler space is a kind of a flat spacetime. One can also generalize these calculation to a D -dimensional black hole geometry (D -dimensional Rindler space). In summary, the claims are: (i) the entropy of the matter is a quantum correction to the Bekenstein-Hawking entropy; (ii) the free energy takes the form

$$\beta F_m(\beta) = \frac{A_D}{\epsilon^{D-2}} C(\beta), \quad (3.3)$$

where A_D is the area of the horizon of a D -dimensional black hole, ϵ is a short-distance cutoff and $C(\beta)$ is a function of β ; (iii) the entropy takes a form similar to (3.3), in particular at $\beta = \beta_H$ it takes the form

$$S_m(\beta_H) \propto \frac{A_D}{\epsilon^{D-2}};$$

and thus (iv) the entropy is divergent owing to the physics at short distance and/or the horizon. Recall the form of the Bekenstein-Hawking entropy $S_{BH} = A_D/(4G_D)$ where G_D

is the gravitational coupling constant in D -dimensions. Then we notice that if we define the ‘‘renormalized gravitational coupling constant’’ by

$$\frac{1}{G_R} = \frac{1}{G_D} + \frac{c}{\epsilon^2},$$

where c is some constant, one can write the total entropy in the same form as S_{BH} :

$$S_{BH} + S_m(\beta_H) = \frac{A_D}{4G_R}.$$

From this observation, Susskind and Uglum argued [38] that (i) the divergence of the entropy is closely related to the renormalization of the gravitational coupling constant, (ii) one can deal with this divergence properly in string theory and (iii) the divergence might store an infinite amount of information.

However, these arguments are based on the results in the Rindler limit. We would like to investigate whether these are still valid for the finite mass cases, i.e., for real black holes. In addition, the discussions are somewhat formal on boundary conditions and regularization schemes. Since a system of a four dimensional black hole plus matter is complicated, it is difficult to discuss these issues in an explicit manner. One often encounters such problems in the study of quantum black holes. Without settling these problems, one cannot reach any definite conclusions. Thus it is very important to make precise analyses as a starting point for getting reliable results. In what follows, we will discuss the thermodynamics of scalar fields in the BTZ black hole geometry to examine these issues [35]. It turns out that we can make definite arguments without ambiguity.

3.2 Thermodynamics of scalar fields by summation over states

In this section, we consider the thermodynamics of scalar fields in the BTZ black hole background by mode expansion and direct computation of summation over states [35]. In order to study the dependence of the thermodynamic quantities upon boundary conditions, we consider two cases. In both cases, we require that the scalar field vanishes rapidly enough at spatial infinity. In one case, we further impose regularity at the origin. In the other case, we impose the condition that the scalar field vanishes near the outer horizon. This is an analog of the one in the previous brick wall model [47, 37, 38]. Although it is possible to consider other various boundary conditions, we do not consider them because their physical meaning is not clear in most cases.

3.2.1 Mode functions

Let us consider a scalar field with mass squared m^2 in a BTZ black hole background. The field equation is given by

$$(\square - \mu l^{-2}) f(x) = 0. \quad (3.4)$$

Since $R = -6l^{-2}$, $\mu l^{-2} = m^2$ for a scalar field minimally coupled to the background metric and $\mu l^{-2} = m^2 + (1/8)R = m^2 - 3/4l^{-2}$ for a conformally coupled scalar field. To solve the equation, it is useful to use the coordinate system $(\hat{r}, \hat{t}, \hat{\varphi})$. Then we expand the field as

$$f(r, t, \varphi) = \sum_{N \in \mathbf{Z}} \int dE f_{EN}(r) e^{-iEt} e^{iN\varphi} = \sum_{\hat{E}, \hat{N}} f_{\hat{E}\hat{N}}(\hat{r}) e^{-i\hat{E}\hat{t}} e^{i\hat{N}\hat{\varphi}}, \quad (3.5)$$

where

$$l^2 E = r_+ \hat{E} + r_- \hat{N}, \quad lN = r_- \hat{E} + r_+ \hat{N}.$$

In the latter expansion, the field equation reads as

$$(\hat{r}^2 - l^2) \partial_{\hat{r}}^2 f_{\hat{E}\hat{N}} + (3\hat{r} - \frac{l^2}{\hat{r}}) \partial_{\hat{r}} f_{\hat{E}\hat{N}} + (\frac{l^2 \hat{E}^2}{\hat{r}^2 - l^2} - \frac{l^2 \hat{N}^2}{\hat{r}^2} - \mu l^{-2}) f_{\hat{E}\hat{N}} = 0.$$

This equation has three regular singular points at $\hat{r} = 0, 1, \infty$ ($r = r_-, r_+, \infty$) corresponding to the inner horizon, the outer horizon and the spatial infinity, respectively. Thus the solution is given by the hypergeometric function. To see this, we make further changes of variables, $u = 1 - \hat{r}^2/l^2$ and

$$f_{\hat{E}\hat{N}}(u) = (-u)^{i\hat{E}/2} (1-u)^{-i\hat{N}/2} g_{\hat{E}\hat{N}}(u).$$

Consequently, the field equation is reduced to the hypergeometric equation [35]

$$u(1-u) \partial_u^2 g_{\hat{E}\hat{N}} + \{c - (a+b+1)u\} \partial_u g_{\hat{E}\hat{N}} - ab g_{\hat{E}\hat{N}} = 0,$$

where

$$\begin{aligned} a &= \frac{1}{2}(1 + \sqrt{1 + \mu}) + i(\hat{E} - \hat{N})/2, \\ b &= \frac{1}{2}(1 - \sqrt{1 + \mu}) + i(\hat{E} - \hat{N})/2, \\ c &= 1 + i\hat{E}. \end{aligned} \quad (3.6)$$

The hypergeometric equation has two independent solutions around each regular singular point. The independent solutions around $u = \infty$ are

$$\begin{aligned} U_{\hat{E}\hat{N}} &= (-u)^{i\hat{E}/2} (1-u)^{-i\hat{N}/2} (-u)^{-a} F(a, a-c+1, a-b+1; 1/u), \\ V_{\hat{E}\hat{N}} &= (-u)^{i\hat{E}/2} (1-u)^{-i\hat{N}/2} (-u)^{-b} F(b, b-c+1, b-a+1; 1/u), \end{aligned} \quad (3.7)$$

where F is the hypergeometric function. In order to specify the solution, we have to impose boundary conditions. First, notice that $U_{\hat{E}\hat{N}}$ vanishes as $r \rightarrow \infty$ for an arbitrary μ , but $V_{\hat{E}\hat{N}}$ becomes divergent there for $\mu > 0$. Second, AdS_3 has timelike spatial infinity

and requires special boundary conditions there. The authors of [49]-[51] have discussed the quantization of scalar fields in anti-de Sitter spaces or their covering spaces in various dimensions. If we follow them and require the Cauchy problem to be well defined, the surface integral of the energy momentum tensor at spatial infinity must vanish:

$$\lim_{r \rightarrow \infty} \int dS_i \sqrt{-g_{BH}} T^i_t = 0. \quad (3.8)$$

This means $\sqrt{r} f_{\hat{E}\hat{N}} \rightarrow 0$ ($r \rightarrow \infty$). Third, in chapter 3, we deal with the case $\mu \geq -3/4$, i.e., the case of non-negative mass squared for both minimally and conformally coupled scalar.⁶ Then, only $U_{\hat{E}\hat{N}}$ satisfies the condition (3.8). Therefore we choose the solution $f_{\hat{E}\hat{N}} = U_{\hat{E}\hat{N}}$, which vanishes rapidly enough at spatial infinity for arbitrary μ .

The above mode functions have been obtained independently of [35] by Ghoroku and Larsen [52]. Using these modes, they have discussed the tachyon scattering in the string theory in the three dimensional black hole geometry. In their case, $\mu = -24/23$. As discussed later, these mode functions are closely related to the representation theory of $SL(2, R)$.

3.2.2 Case I

To examine the dependence of the thermodynamic quantities on boundary conditions, we will consider two further boundary conditions. For usual radial functions in quantum mechanics, one requires square integrability, and this condition leads to regularity at the origin. In a BTZ background, the meaning of the square integrability is not clear until we specify the inner product. However, we do not know what inner product we should adopt. Thus as an exercise for the above purpose, we first impose on $f_{\hat{E}\hat{N}}$ regularity at the origin ($r = 0$).

It is easy to solve this boundary condition. Indeed, we readily find that $U_{\hat{E}\hat{N}}$ is regular at the origin because $r = 0$ corresponds to none of $\hat{r}/l = 0, 1, \infty$. Thus we need no restriction on the value of E .

Here let us recall that a system of a rotating black hole plus a scalar field has a chemical potential Ω_H . This is the angular velocity of the outer horizon:

$$\Omega_H = -N^\varphi(r_+) = \frac{r_-}{lr_+}.$$

In addition, the system has superradiant scattering modes given by the condition

$$E - \Omega_H N \leq 0,$$

⁶However, since we can deal with the case $-3/4 > \mu \geq -1$ on the same footing, we will include this case in the following discussion.

where E and N are the energy and angular momentum of the scalar field, respectively. Thus we have to regularize the (grand) partition function by introducing a cutoff Λ_1 for the occupation number of particles for each superradiant scattering mode.

With this remark in mind, the remaining calculation is straightforward and, by introducing appropriate cutoffs, we will obtain explicit results [35]. First, the partition function for a single mode labeled by E, N and the inverse temperature β is give by⁷

$$Z_o(\beta; E, N) = \sum_{n=0}^{\infty} e^{-n\beta(E-\Omega_H N)} = \begin{cases} \left(1 - e^{-\beta(E-\Omega_H N)}\right)^{-1} & \text{for } E - \Omega_H N > 0 \\ \Lambda_1 & \text{for } E - \Omega_H N = 0 \\ \frac{1 - e^{-\Lambda_1\beta(E-\Omega_H N)}}{1 - e^{-\beta(E-\Omega_H N)}} & \text{for } E - \Omega_H N < 0 \end{cases}.$$

Then we obtain the total partition function,

$$Z_o(\beta) = \prod_{E, N} Z_o(\beta; E, N),$$

and the free energy,

$$\begin{aligned} -\beta F_o(\beta) &= \sum_{E, N} \ln Z_o(\beta; E, N) \\ &= - \sum_{|N|=0}^{\Lambda_2} \frac{1}{s} \int_0^\infty dE \ln \left(1 - e^{-\beta(E-\Omega_H N)}\right) + \sum_{N=0}^{\Lambda_2} \Lambda_1 \\ &\quad + \sum_{N=0}^{\Lambda_2} \frac{1}{s} \int_0^{N\Omega_H} dE \ln \left(1 - e^{-\beta\Lambda_1(E-\Omega_H N)}\right), \end{aligned}$$

where Λ_2 is the cutoff for the absolute value of quantum number N , and s is the minimum spacing of E . Note that s^{-1} is the density of states and the above result is divergent as $s \rightarrow 0$ regardless of the existence of the horizon. Furthermore, by making the change of variables $y = \beta(E - \Omega_H N)$ for the first term and $y = \Lambda_1\beta(N\Omega_H - E)$ for the third term, we obtain

$$\begin{aligned} -\beta F_o(\beta) &= \frac{1}{s} \left[\frac{\pi^2}{6\beta} (2\Lambda_2 + 1) + \frac{\beta}{12} \Omega_H^2 (\Lambda_1 - 1) \Lambda_2 (\Lambda_2 + 1) (2\Lambda_2 + 1) \right. \\ &\quad \left. + \frac{1}{\Lambda_1\beta} \sum_{N=1}^{\Lambda_2} \int_0^{\Lambda_1\beta\Omega_H N} dy \ln \left(1 - e^{-y}\right) \right] + \Lambda_1 (\Lambda_2 + 1). \end{aligned}$$

⁷The value $E = \Omega_H N$ is similar to the value where Bose condensation occurs. Then we might have to omit the summation over the superradiant scattering modes because they may be a signal for instability. However, we can get the quantities omitting this summation simply by setting $\Lambda_1 = 0$.

In the limit $\Lambda_1 \rightarrow \infty$, the last term in the bracket is simplified to $-\Lambda_2 \zeta(2)/(\Lambda_1 \beta)$. Finally, by the formula (3.2), we get the entropy

$$\begin{aligned} S_o(\beta) = & \frac{1}{s} \left[\frac{\pi^2}{3\beta} (2\Lambda_2 + 1) - \Omega_H \sum_{N=1}^{\Lambda_2} N \ln \left(1 - e^{-\Lambda_1 \beta \Omega_H N} \right) \right. \\ & \left. + \frac{2}{\Lambda_1 \beta} \sum_{N=1}^{\Lambda_2} \int_0^{\Lambda_1 \beta \Omega_H N} dy \ln \left(1 - e^{-y} \right) \right] + \Lambda_1 (\Lambda_2 + 1). \end{aligned}$$

Thus, the entropy is not proportional to the area (perimeter) of the outer horizon ($2\pi r_+$) for a generic β including $\beta = \beta_H$ in (2.6). In addition, its divergence is not due to the existence of the outer horizon. It comes from the cutoff s .

3.2.3 Case II

Let us consider another boundary condition. One may expect that something singular occurs at the horizon since, e.g., the redshift factor of the black hole becomes divergent there. Thus, as the second case, we require regularity at the outer horizon. This boundary condition is an analog of the one in the brick wall model [47, 37, 38]. In our case, we can solve this condition and obtain thermodynamic quantities without the WKB approximation [35].

To solve the boundary condition, we first study the behavior of $U_{\hat{E}\hat{N}}$ near the outer horizon ($r = r_+$, i.e., $\hat{r}/l = 1$). By making use of a linear transformation formula with respect to the hypergeometric function, we get

$$\begin{aligned} U_{\hat{E}\hat{N}} \propto & (-u)^{i\hat{E}/2} (1-u)^{-i\hat{N}/2} F(a, b; c; u) \\ & - \Theta (-u)^{-i\hat{E}/2} (1-u)^{-i\hat{N}/2} F(a-c+1, b-c+1; 2-c; u), \end{aligned}$$

where

$$\Theta = \frac{\Gamma(1-b)\Gamma(c)\Gamma(a-c+1)}{\Gamma(a)\Gamma(2-c)\Gamma(c-b)}.$$

From $(\Gamma(z))^* = \Gamma(z^*)$ and (3.6), we find that $|\Theta| = 1$ for $\mu \geq -1$. Thus we may set

$$\Theta = e^{-2\pi i \theta_0} \quad (0 \leq \theta_0 < 1),$$

where θ_0 is determined by E and N through a , b and c . Choosing an appropriate normalization constant, we can write

$$\begin{aligned} U_{\hat{E}\hat{N}} = & (-u)^{i\hat{E}/2} (1-u)^{-i\hat{N}/2} e^{\pi i \theta_0} F(a, b; c; u) \\ & - (-u)^{-i\hat{E}/2} (1-u)^{-i\hat{N}/2} e^{-\pi i \theta_0} F(a-c+1, b-c+1; 2-c; u). \end{aligned}$$

Then by introducing an infinitesimal constant ϵ_H and substituting $-u = \epsilon_H^2/l^2$ (namely, $\epsilon_H^2/l^2 = (r^2 - r_+^2)/d_H^2$), we find the behavior of $U_{\hat{E}\hat{N}}$ near the outer horizon;

$$U_{\hat{E}\hat{N}} \xrightarrow{\epsilon_H/l \rightarrow 0} e^{i\hat{E}\ln(\epsilon_H/l) + \pi i\theta_0} = e^{-(i\hat{E}\ln(\epsilon_H/l) + \pi i\theta_0)}.$$

Here, we impose the boundary condition $U_{\hat{E}\hat{N}} = 0$ at $-u = \epsilon_H^2/l^2$. This condition yields

$$E = \Omega_H N + C(K + \theta_0) \quad (K \in \mathbf{Z}), \quad C = \frac{\pi d_H^2}{r_+ l^2 \ln(l/\epsilon_H)}.$$

This shows that E and θ_0 are labeled by two integers K and N , i.e., $E = E(K, N)$ and $\theta_0 = \theta_0(K, N)$. C^{-1} becomes singular as $\epsilon_H \rightarrow 0$. In this way, we could solve the “brick wall” boundary condition without the WKB approximation.

Then, we can obtain the partition function and the free energy similarly to the previous case:

$$\begin{aligned} -\beta F_h(\beta) &= \sum_{K, N} \ln Z_h(\beta; K, N) \\ &= - \sum_{|N|=0}^{\Lambda_2} \sum_{\substack{K+\theta_0 \neq 0 \\ C(K+\theta_0) \geq -\Omega_H N}} \ln \left(1 - e^{-C\beta(K+\theta_0)} \right) + \Lambda_1 \sum_{N=0}^{\Lambda_2} \delta_{\theta_0(0, N), 0} \\ &\quad + \sum_{N=1}^{\Lambda_2} \sum_{0 > C(K+\theta_0) \geq -\Omega_H N} \ln \left(1 - e^{-\Lambda_1 C\beta(K+\theta_0)} \right). \end{aligned}$$

Since $C \ll 1$ in the limit $\epsilon_H/l \rightarrow 0$, the summation with respect to K can be approximated by integrals. Notice that

$$\frac{dK}{dE} = \frac{1}{C} - \frac{d\theta_0}{dK} \frac{dK}{dE} \sim \frac{1}{C}.$$

This shows that the density of states diverges because of the existence of the outer horizon, i.e., as $\epsilon_H/l \rightarrow 0$. In this limit, we get

$$-\beta F_h(\beta) \sim \frac{r_+ s l^2 \ln(l/\epsilon_H)}{\pi d_H^2} \{-\beta F_o(\beta) - \Lambda_1(\Lambda_2 + 1)\} + \Lambda_1 \sum_{N=0}^{\Lambda_2} \delta_{\theta_0(0, N), 0}.$$

Then the entropy is

$$S_h(\beta) \sim \frac{r_+ s l^2 \ln(l/\epsilon_H)}{\pi d_H^2} \{S_o(\beta) - \Lambda_1(\Lambda_2 + 1)\} + \Lambda_1 \sum_{N=0}^{\Lambda_2} \delta_{\theta_0(0, N), 0}.$$

Therefore, (i) the leading term of the entropy as $\epsilon_H/l \rightarrow 0$ is proportional to $r_+ l^2/(d_H^2 \beta)$, and (ii) the entropy diverges owing to the outer horizon like $\ln(l/\epsilon_H)$ as $\epsilon_H/l \rightarrow 0$. Thus again the entropy is not proportional to the area for a generic β including $\beta = \beta_H$.

In section 3.2, we calculated the thermodynamic quantities in a truly curved spacetime without the heat kernel expansion nor the WKB approximation. This enables us to make definite discussions on the thermodynamics of a scalar field. From the results, we find that the physical quantities depend largely upon boundary conditions as expected. In case II where the boundary condition is closely related to the horizon, the thermodynamic quantities are expressed by the parameters related to the horizon, e.g., r_+ and ϵ_H . But they are not so in case I. The divergence of the entropy is due to the horizon in case II, but this is not so in case I. These indicate that one must justify the brick wall boundary condition if one adopts it. Otherwise the quantities obtained in that manner may be artifacts of the boundary condition. The entropies are not proportional to the area of the horizon. This may be related to the special properties of the BTZ black hole. The BTZ black hole has a fundamental scale l and asymptotically AdS_3 instead of Minkowski spacetime. However, the naive expansion in the literature with respect to the inverse black hole mass may not converge, and the entropy may receive a large correction. Thus further investigations in a truly curved case may be necessary.

3.3 Green functions in the three dimensional black hole background

Now, let us turn to a new approach to the thermodynamics using Green functions. Comparing the results with the previous ones, we can examine the equivalence among various methods for the thermodynamics of a scalar field in a black hole geometry. In the literature, Green functions are evaluated by the heat kernel expansion. However in our case, we can construct the exact Green functions for scalar fields with a generic mass squared [35]. Making use of them, we can obtain exact expressions of the thermodynamic quantities [35]. As a first step, we will discuss the Green functions in this section.

3.3.1 Construction of Green functions

For a conformally coupled massless scalar in a BTZ background, the Green functions have been obtained in [42, 43] by making use of the Green functions in AdS_3 and of the method of images. Here we will generalize this construction to a generic mass squared.

Quantization of a scalar field in the universal covering space of D -dimensional anti-de Sitter space (\widetilde{AdS}_D) has been discussed in [49]-[51]. For a generic value of D , the Feynman Green function is given in terms of the hypergeometric functions [51]. But for ($D = 3$), it is expressed by a simple form [35]

$$-iG_F(x, x') = -iG_F(z) \equiv \frac{1}{4\pi l} (z^2 - 1)^{-1/2} \left[z + (z^2 - 1)^{1/2} \right]^{1-\lambda}, \quad (3.9)$$

where

$$\begin{aligned} z &= 1 + l^{-2} \sigma(x, x') + i\varepsilon, \\ \lambda &= \begin{cases} \lambda_{\pm} \equiv 1 \pm \sqrt{1+\mu} & \text{for } 0 > \mu > -1, \\ \lambda_{+} & \text{for } \mu \geq 0, \mu = -1, \end{cases} \end{aligned}$$

(two λ 's are possible for $0 > \mu > -1$) and ε is a positive and infinitesimal constant. $\sigma(x, x')$ is half of the distance between x and x' in the four dimensional embedding space,

$$\sigma(x, x') = \frac{1}{2} \eta_{\alpha\beta} (x - x')^{\alpha} (x - x')^{\beta},$$

where $\eta_{\alpha\beta}$ and x^{α} ($\alpha, \beta = 0-3$) are given by (2.1) and (2.2). Since the derivation of this result is technical, we relegate it to appendix A.

By making use of the above result, Green functions for a generic mass squared in a BTZ background are obtained by the method of images [35];

$$\begin{aligned} -iG_{BH}(x, x') &= -i \sum_{n=-\infty}^{\infty} G_F(x, x'_n) \\ &= \frac{1}{4\pi l} \sum_{n=-\infty}^{\infty} (z_n^2 - 1)^{-1/2} [z_n + (z_n^2 - 1)^{1/2}]^{1-\lambda}, \end{aligned}$$

where

$$x_n \equiv x \Big|_{\varphi \rightarrow \varphi - 2n\pi}, \quad z_n(x, x') = z(x, x'_n).$$

From (A.4), we can check

$$(\square - \mu l^{-2})G_{BH} = \frac{1}{\sqrt{-g_{BH}}} \delta(x - x'), \quad (3.10)$$

where $\delta(x - x')$ is the delta function in the black hole geometry. Note that the Green functions are functions of

$$\begin{aligned} z_n(x, x') - i\varepsilon &= \frac{1}{d_H^2} \left[\sqrt{r^2 - r_-^2} \sqrt{r'^2 - r_-^2} \cosh \left(\frac{r_-}{l^2} \Delta t - \frac{r_+}{l} \Delta \varphi_n \right) \right. \\ &\quad \left. - \sqrt{r^2 - r_+^2} \sqrt{r'^2 - r_+^2} \cosh \left(\frac{r_+}{l^2} \Delta t - \frac{r_-}{l} \Delta \varphi_n \right) \right], \quad (3.11) \end{aligned}$$

where

$$\Delta t = t - t', \quad \Delta \varphi_n = \varphi - \varphi' + 2n\pi.$$

For the conformally coupled massless scalar field, namely, for $\mu = -3/4$, we have $\lambda_{\pm} = 3/2, 1/2$, and the Green functions are reduced to

$$-iG_{BH}(x, x') = \frac{1}{2^{\lambda+1} \pi l} \sum_{n=-\infty}^{\infty} \left[\frac{1}{\sqrt{z_n - 1}} \pm \frac{1}{\sqrt{z_n + 1}} \right].$$

These coincide with the Green functions discussed in [42, 43] which have the “Neumann” or the “Dirichlet” boundary condition.

3.3.2 Boundary conditions and the vacuum

We have constructed the Green functions in a BTZ black hole background. However, the physical meaning of the Green functions is not clear unless we specify its boundary conditions and identify the vacuum with respect to which they are defined. It turns out [35] that G_{BH} satisfies the boundary conditions: (i) to be regular at infinity, (ii) to be analytic in the upper half plane on the past complexified outer horizon, and (iii) to be analytic in the lower half plane on the future complexified outer horizon. These conditions fix G_{BH} as a solution of the inhomogeneous wave equation [8]. This means that the vacuum is defined by the Kruskal modes, i.e., it is the Hartle-Hawking vacuum [8, 9]. Thus the Green function is regarded as the Hartle-Hawking Green function which is important to discussions on thermodynamics of black holes and Hawking radiation.

For the conformally coupled massless scalar field ($\mu = -3/4$) in a non-rotating black hole geometry ($J_{BH} = 0$), the statements (i)-(iii) have been verified [43]. Thus we follow the strategy in [43]. For brevity, we concentrate on the case $r, r' \geq r_+$ in the following.

First, from the definition of G_{BH} , we easily find that the boundary condition (i) is satisfied .

Let us turn to the condition (ii). We first introduce Kruskal coordinates [31] by

$$\begin{aligned} V &= R(r) e^{a_H t}, & U &= -R(r) e^{-a_H t}, \\ R(r) &= \sqrt{\left(\frac{r-r_+}{r+r_+}\right) \left(\frac{r+r_-}{r-r_-}\right)^{r_-/r_+}}, \end{aligned} \quad (3.12)$$

where

$$a_H = \frac{r_+^2 - r_-^2}{r_+ l^2} = \frac{2\pi}{\beta_H}.$$

In this coordinate system, the metric becomes

$$\begin{aligned} ds_{BH}^2 &= \Omega^2(r) dU dV + r^2 (N^\varphi dt + d\varphi)^2, \\ \Omega^2(r) &= \frac{r_+^2 (r^2 - r_-^2) (r + r_+)^2}{d_H^4 r^2} \left(\frac{r-r_-}{r+r_+}\right)^{r_-/r_+}. \end{aligned}$$

From (3.12), the past complexified outer horizon is given by $V = 0$ and $\text{Re}(-U) > 0$. In terms of (t, r) , this reads as

$$\begin{cases} r \longrightarrow r_+ \\ t \longrightarrow -\infty, \end{cases} \quad \text{with} \quad \sqrt{r-r_+} e^{-a_H t} \longrightarrow \gamma_H.$$

Here γ_H is a constant determined by the value of U and has the property $\text{Re} \gamma_H > 0$. $\text{Im} \gamma_H > 0$ and $\text{Im} \gamma_H < 0$ correspond to the lower and the upper half plane of U , respectively.

Then let us examine the regularity of G_{BH} . For this purpose, we further introduce a new angle coordinate rotating together with the outer horizon;

$$\varphi^+ = \varphi - \Omega_H t.$$

Note that $N^\varphi dt + d\varphi = d\varphi^+$ on the outer horizon. This is an analog of the angle coordinate in the Kerr geometry which is used for obtaining the regular expression of the metric on the outer horizon and for maximally extending the spacetime [53]. Using φ^+ , z_n are written as

$$\begin{aligned} z_n(x, x') - i\varepsilon &\equiv z_n(w_n, \Delta\varphi_n^+; r, r') - i\varepsilon \\ &= \frac{1}{d_H^2} \left[\sqrt{r_+^2 - r_-^2} \sqrt{r'^2 - r_-^2} \cosh \left(\frac{r_+}{l} \Delta\varphi_n^+ \right) - \sqrt{r^2 - r_+^2} \sqrt{r'^2 - r_+^2} \cosh(iw_n) \right], \end{aligned} \quad (3.13)$$

where

$$\begin{aligned} \Delta\varphi_n^+ &= \varphi^+ - \varphi'^+ + 2n\pi, \\ iw_n &= a_H \Delta t - \frac{r_-}{l} \Delta\varphi_n^+. \end{aligned}$$

Thus, on the past complexified outer horizon we have

$$\begin{aligned} z_n(x, x') - i\varepsilon &\longrightarrow \frac{1}{d_H^2} \left[\sqrt{r_+^2 - r_-^2} \sqrt{r'^2 - r_-^2} \cosh \left(\frac{r_+}{l} \Delta\varphi_n^+ \right) \right. \\ &\quad \left. - \sqrt{r_+ l/2} \sqrt{r'^2 - r_+^2} e^{a_H t' + (r_-/l) \Delta\varphi_n^+} \gamma_H \right]. \end{aligned}$$

Let us recall that each component in the summation in G_{BH} has singularities at $z_n = \pm 1$. From the above expression of z_n , we find that the points $z_n = \pm 1$ on the past complexified outer horizon correspond to

$$\gamma_H = \alpha_0^\pm + i\varepsilon,$$

where α_0^\pm are some positive numbers. Consequently, each component in G_{BH} is regular in the upper half plane of U .

Then we will use Weierstrass' theorem [54]: if a series with analytic terms converges uniformly on every compact subset of a region, then the sum is analytic in that region, and the series can be differentiated term by term. Since the series in G_{BH} converges uniformly, G_{BH} is analytic in the upper half plan of the past complexified outer horizon. This completes the proof of (ii).

The proof of (iii) is similar, and we omit it.

The Hartle-Hawking Green function in a black hole geometry was originally defined in the path-integral formalism as a generalization of the Feynman Green function in Minkowski spacetime [8]. In our case, G_F is also defined so as to conform to the Feynman Green function in the flat limit (see (A.5) and the comment below it). Thus it is natural for G_F to satisfy the Hartle-Hawking boundary condition.

3.4 Thermodynamics of scalar fields by Hartle-Hawking Green functions

We have constructed the Hartle-Hawking Green functions in the BTZ background. In this section, we discuss thermodynamics of a scalar field by using them [35].

3.4.1 Euclidean Green functions

For calculating the thermodynamic quantities, we will introduce Euclidean Green functions and study their properties [35]. First, we define the Euclidean time by $\tau = it$ and the “Euclidean angle” by $\varphi_E = -i\varphi$ for $J_{BH} \neq 0$ and $\varphi_E = \varphi$ for $J_{BH} = 0$. Furthermore, we introduce a new coordinate

$$i\Delta\varphi_n^{E+} \equiv \Delta\varphi_n^+ = \begin{cases} i(\Delta\varphi_E + \Omega_H\Delta\tau) + 2\pi n & \text{for } J_{BH} = 0 \\ \Delta\varphi_E + i\Omega_H\Delta\tau + 2\pi n & \text{for } J_{BH} \neq 0 \end{cases}.$$

Then we have

$$-w_n = a_H\Delta\tau + \frac{r_-}{l}\Delta\varphi_n^{E+}. \quad (3.14)$$

Using the Euclidean coordinates, we obtain the Euclidean black hole geometry

$$ds_E^2 = g_{\mu\nu}^E dx^\mu dx^\nu = \begin{cases} N_\perp^2 d\tau^2 + r^2 d\varphi_E^2 + N_\perp^{-2} dr^2 & \text{for } J_{BH} = 0 \\ N_\perp^2 d\tau^2 - r^2 (N^\varphi d\tau - d\varphi_E)^2 + N_\perp^{-2} dr^2 & \text{for } J_{BH} \neq 0 \end{cases}.$$

Notice that the Euclidean metric becomes complex unless we use the Euclidean angle.⁸ Then the Green function in the Euclidean geometry (the Euclidean Green function) is given by

$$\begin{aligned} G_{BH}^E(x, x') &\equiv \sum_{n=-\infty}^{\infty} G_F^E(x, x'_n), \\ G_F^E(x, x'_n) &\equiv iG_F(z(w_n(\Delta\tau, \Delta\varphi_n^{E+}), i\Delta\varphi_n^{E+}; r, r')), \end{aligned} \quad (3.15)$$

(recall (3.13)). The factor in front of G_F was chosen so that the physical quantities calculated later will have real values and appropriate signs. From (3.10), we can check

$$(\square^E - \mu l^{-2})G_{BH}^E = \frac{a}{\sqrt{|g^E|}} \delta^E(x - x'),$$

where $a = -1$ for $J_{BH} = 0$ and $a = i$ for $J_{BH} \neq 0$.

Next, we consider thermal properties of G_{BH}^E . The Green function G_{BH} was a function of z_n given by (3.11). Thus G_{BH}^E for $J_{BH} \neq 0$ is periodic under

$$\begin{aligned} \delta\left(r_- \frac{\tau}{l^2} + r_+ \frac{\varphi}{l}\right) &= 2\pi m \\ \delta\left(r_+ \frac{\tau}{l^2} + r_- \frac{\varphi}{l}\right) &= 2\pi n \quad (m, n \in \mathbf{Z}), \end{aligned}$$

⁸We can make the Euclidean metric real by a continuation $J_{BH}^E = -iJ_{BH}$ instead of $\varphi_E = -i\varphi$ [32]. The following discussions are valid also in this case with slight modifications.

where $\delta(\dots)$ means the variation of the arguments. Namely, G_{BH}^E is of double period

$$\begin{pmatrix} \delta(\tau/l) \\ \delta\varphi \end{pmatrix} = \frac{2\pi l}{d_H^2} \begin{pmatrix} -r_- & r_+ \\ r_+ & -r_- \end{pmatrix} \begin{pmatrix} m \\ n \end{pmatrix}.$$

If we require that the chemical potential vanishes as $J_{BH} \rightarrow 0$ ($r_- \rightarrow 0$), the fundamental period is determined uniquely as

$$\tau \rightarrow \tau + \beta_H, \quad \varphi \rightarrow \varphi - \nu_{BH}\beta_H,$$

where

$$\nu_{BH} = \frac{r_-}{lr_+} = \Omega_H.$$

This periodicity is also valid for $J_{BH} = 0$. Recall that a thermal Green function at temperature β^{-1} and with a chemical potential ν conjugate to angular momentum is defined by

$$G_\beta^E(x, x'; \nu) = \frac{\text{Tr} \left[e^{-\beta(\hat{H}-\nu\hat{L})} T(\psi(x)\psi(x')) \right]}{\text{Tr} \left[e^{-\beta(\hat{H}-\nu\hat{L})} \right]},$$

where T denotes the (Euclidean) time ordered product and \hat{H} and \hat{L} are the generators of time translation and rotation, respectively. From the definition, we have

$$G_\beta^E(\tau, \varphi, r; \tau', \varphi', r'; \nu) = G_\beta^E(\tau + \beta, \varphi - \nu\beta, r; \tau', \varphi', r'; \nu).$$

Comparing this with G_{BH}^E , G_{BH}^E can be regarded as a thermal Green function with the inverse temperature $\beta = \beta_H$ and the chemical potential ν_{BH} . This is consistent with the classical result in (2.6). In the following, we will explicitly denote the period of Green functions, for example, as $G_{BH}^E(x, x'; \beta_H)$.

It is instructive to consider the behavior of the metric near the outer horizon. We introduce a coordinate η by

$$r = r_+ + \frac{2}{a_H} \eta^2.$$

Then, for small η , the metric becomes

$$ds^2 \sim -(2\pi/\beta_H)^2 \eta^2 dt^2 + d\eta^2 + r_+^2 (d\varphi^+)^2. \quad (3.16)$$

In terms of the Euclidean time $\tau = it$, (τ, r) (or (τ, η)) space represents a plane with the origin $r = r_+$ ($\eta = 0$). Therefore β_H is nothing but the period around the outer horizon of the Euclidean black hole. The periodicity of G_{BH}^E gives an explicit example to the arguments in the literature of thermodynamics of black holes.

3.4.2 Free energy

By making use of the Euclidean Green function, we calculate the free energy in this subsection. In terms of the Euclidean Green function, the free energy is given by the formula (3.2). In our case, the trace is defined by

$$\begin{aligned}\text{Tr}(\dots) &= \int d^3x \sqrt{|g^E|} \lim_{x \rightarrow x'} (\dots) \\ &= \begin{cases} \int_0^\beta d\tau \int_0^{2\pi} d\varphi_E \int_{r_+}^\infty dr \cdot r \lim_{x \rightarrow x'} (\dots) & \text{for } J_{BH} = 0 \\ \int_0^\beta d\tau \int_0^{\Omega_H \beta} d\varphi_E \int_{r_+}^\infty dr \cdot r \lim_{x \rightarrow x'} (\dots) & \text{for } J_{BH} \neq 0 \end{cases}.\end{aligned}$$

Here we have set the lower end of the integration with respect to r to be r_+ . The reasons are (i) in the Euclidean geometry, the topology of (τ, r) space is \mathbf{R}^2 and the origin corresponds to $r = r_+$, and (ii) it turns out that the entropy becomes complex if we perform integration below r_+ .

For flat spacetime, an expression for free energy like (3.2) is divergent, and we have to regularize it by differentiating it with respect to mass squared. To get the right answer, we then integrate the differentiated expression. Thus we will follow the prescription for flat spacetime. In our case, the parameter μ corresponds to mass squared and we get

$$\frac{\partial}{\partial \mu} (\beta F(\beta)) = -\frac{1}{2l^2} \text{Tr } G_{BH}^E(\beta).$$

where we have used $G_{BH}^E = (\square^E - \mu l^{-2})^{-1}$ (up to a factor).

Then the remaining calculation is straightforward. We first consider $J_{BH} \neq 0$ case. In this case, we have

$$\frac{\partial}{\partial \mu} (\beta F(\beta)) \Big|_{\beta=\beta_H} = -i \frac{\Omega_H}{4l^2} \beta^2 \sum_{n=-\infty}^{\infty} \int_{r_+^2}^{\infty} d(r^2) \lim_{r \rightarrow r'} G_F^E(z_n^0; \beta_H) \Big|_{\beta=\beta_H}, \quad (3.17)$$

where $z_n^0 = z_n|_{\Delta\tau=\Delta\varphi_E=0}$, and we have used the fact that the integrand is independent of τ and φ_E . Recall the expression of G_F and z_n , i.e., (3.9) and (3.11). Then the integrand with $n = 0$ in the summation diverges as $r \rightarrow r'$. So we remove this term for a moment. Notice that $G_F(z_n; \beta_H)$ and z_n^0 are written as

$$\begin{aligned}-iG_F(z_n; \beta_H) &= \begin{cases} \frac{1}{4\pi l(1-\lambda)} \frac{d}{dz_n} e^{(1-\lambda)\coth^{-1} z_n} & \text{for } \lambda \neq 1 \\ \frac{1}{4\pi l} (z_n^2 - 1)^{-1/2} & \text{for } \lambda = 1 \end{cases}, \\ z_n^0 \Big|_{r=r'} &= \frac{1}{d_H^2} \left\{ (r^2 - r_-^2) c_n^+ - (r^2 - r_+^2) c_n^- \right\},\end{aligned}$$

where

$$c_n^\pm = \cosh \left(2\pi n \frac{r_\pm}{l} \right).$$

Here and in the following, we omit the infinitesimal imaginary part of z_n except when it is relevant for the discussion. Then by making the change of variables from r^2 to z_n^0 , we get

$$\begin{aligned} & -i \int_{r_+^2}^{\infty} d(r^2) \lim_{r \rightarrow r'} G_F^E(z_n^0; \beta_H) \\ &= \begin{cases} \frac{1}{4\pi l} \frac{d_H^2}{(c_n^+ - c_n^-)(1 - \lambda)} (z + \sqrt{z^2 - 1})^{1-\lambda} \Big|_{c_n^+}^{\infty} & \text{for } \lambda \neq 1 \\ \frac{1}{4\pi l} \frac{2d_H^2}{(c_n^+ - c_n^-)} \ln(z + \sqrt{z^2 - 1}) \Big|_{c_n^+}^{\infty} & \text{for } \lambda = 1 \end{cases}. \end{aligned}$$

The integral diverges at the upper end for $\lambda \leq 1$, but for $\lambda > 1$ we get

$$\frac{\partial}{\partial \mu} (\beta F(\beta)) \Big|_{\beta=\beta_H} = \frac{\Omega_H \beta_H^2 d_H^2}{8\pi l^3 (\lambda - 1)} \sum_{n \geq 1}^{\infty} \frac{1}{c_n^+ - c_n^-} e^{-2\pi(\lambda-1)nr_+/l} + c_0,$$

where c_0 is the divergent term coming from $n = 0$. Consequently, by integrating the above expression, we obtain [35]

$$\begin{aligned} \beta F(\beta) \Big|_{\beta=\beta_H} &= -\frac{\Omega_H \beta_H^2 d_H^2}{8\pi^2 r_+ l^2} \sum_{n \geq 1}^{\infty} \frac{1}{n(c_n^+ - c_n^-)} e^{-2\pi(\lambda-1)nr_+/l} \\ &\quad + \text{const.} \quad (\text{for } \lambda > 1). \end{aligned}$$

Similarly, we get the result for $J_{BH} = 0$;

$$\begin{aligned} \beta F(\beta) \Big|_{\beta=\beta_H} &= -\frac{r_+ \beta_H}{4\pi l^2} \sum_{n \geq 1}^{\infty} \frac{1}{n(c_n^+ - c_n^-)} e^{-2\pi(\lambda-1)nr_+/l} \\ &\quad + \text{const.} \quad (\text{for } \lambda > 1). \end{aligned}$$

This can be obtained also by the replacement $\Omega_H \beta_H \rightarrow 2\pi$ in the expression for $J_{BH} \neq 0$.

3.4.3 Green functions on a cone geometry

In order to calculate the entropy, we have to differentiate the Euclidean Green function with respect to β with the chemical potential fixed. Namely, we need the Green functions with a period different from β_H with Ω_H fixed. These Green functions are regarded as those on (τ, r) plane with a deficit angle around the origin, i.e., on a cone geometry. In this subsection, we will construct the Green functions on a cone geometry with an arbitrary period [35].

To get them, we first take G_{BH}^E as a function of $r^{(\prime)}$, $\Delta\tau$ and $\Delta\varphi^{E+}$. Then they are obtained by (i) fixing the values of $r^{(\prime)}$ and $\Delta\varphi^{E+}$ and (ii) changing the period with respect to $\Delta\tau$. The chemical potential is surely fixed by this procedure. We denote these Green functions by

$$G_{BH}^E(x, x'; \beta) = \sum_{n=-\infty}^{\infty} G_F^E(x, x'_n; \beta),$$

where β is the period of $\Delta\tau$, and $G_F^E(x, x'_n; \beta)$ are Green functions on a cone geometry corresponding to $G_F^E(x, x'_n; \beta_H)$. $G_{BH}^E(x, x'_n; \beta_H)$ and $G_F^E(x, x'_n; \beta_H)$ are the Green functions in (3.15), which we have considered so far.

The authors of [55] discussed how to construct Green functions with an arbitrary period for certain differential equations. Their method was also applied to field theory on curved spaces [56, 40]. By following them, we can obtain the explicit form of G_F^E [35]:

$$\begin{aligned} G_F^E(x, x'_n; \beta) &= \frac{\beta_H}{2\pi\beta} \int_{\Gamma} d\zeta \tilde{G}_F^E(\zeta; 2\pi) \frac{e^{i\beta_H\zeta/\beta}}{e^{i\beta_H\zeta/\beta} - e^{i\beta_H w_n/\beta}}, \\ &= \frac{\beta_H}{4\pi i\beta} \int_{\Gamma} d\zeta \tilde{G}_F^E(\zeta; 2\pi) \cot \left\{ \frac{\beta_H}{2\beta} (\zeta - w_n) \right\}. \end{aligned}$$

where $\tilde{G}_F^E(\zeta; 2\pi)$ is defined by

$$\tilde{G}_F^E(\zeta; 2\pi) \equiv G_F^E(z(\zeta, \Delta\varphi_n^{E+}; r, r'); \beta_H) \Big|_{\Delta\varphi_n^{E+}, r, r': \text{fixed}}.$$

From (3.13) (or (B.1)), $\tilde{G}_F^E(\zeta; 2\pi)$ is periodic under $\zeta \rightarrow \zeta + 2\pi$ ($\tau \rightarrow \tau + \beta_H$). The contour Γ is given by the solid lines in Figure 2. Making use of this expression, we can get the derivative of the Green function with respect to β . At $\beta = \beta_H$, it is given by [35]

$$\begin{aligned} &\frac{\partial}{\partial\beta} G_F^E(x, x'_n; \beta) \Big|_{\beta=\beta_H} \\ &= -\frac{B}{\beta_H} \int_{A_n+B}^{\infty} dz G_F^E(z; \beta_H) \frac{1}{\sqrt{(z - A_n)^2 - B^2}} \frac{c_n(z - A_n) + B}{(z - A_n + c_n B)^2}, \end{aligned} \quad (3.18)$$

where

$$\begin{aligned} A_n &= \frac{1}{d_H^2} \sqrt{r^2 - r_-^2} \sqrt{r'^2 - r_-^2} \cosh \left(\frac{ir_+}{l} \Delta\varphi_n^{E+} \right), \quad B = \frac{1}{d_H^2} \sqrt{r^2 - r_+^2} \sqrt{r'^2 - r_+^2}, \\ c_n &= \cosh(iw_n). \end{aligned}$$

The derivation of these results is given in appendix B.

Note that $B > 0$ for $r, r' > r_+$ and $B < 0$ for $r, r' < r_+$. Thus, for $r, r' < r_+$, $A_n + B$ can be less than 1, and the Green function $-iG_F^E(z; \beta_H)$ becomes complex. Because of the contribution from this region, the entropy also becomes complex. This indicates that we should consider only the region $r > r_+$ in the calculation of thermodynamic quantities as we have done.

3.4.4 Entropy

Making use of the Green functions with a generic β , we can obtain the entropy [35]. Since the definition of the trace is different between $J_{BH} = 0$ and $J_{BH} \neq 0$ case, we first consider

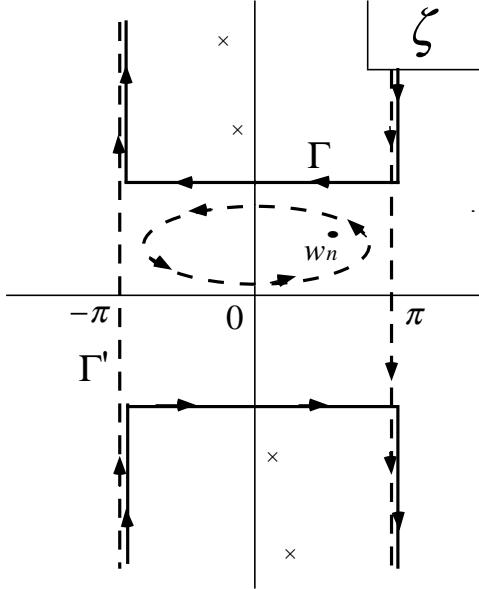


Figure 2 : Contour Γ (solid lines) and Contour Γ' (dashed lines) in ζ plane. The crosses (\times) indicate the singularities of $\tilde{G}_F^E(\zeta; 2\pi)$ in the region $-\pi < \text{Re } \zeta \leq \pi$ for $r, r' \geq r_+$. The dot (\bullet) indicates w_n up to $2\pi m$. In this figure, we show the contour Γ for small $|\text{Im } w_n|$. For large $|\text{Im } w_n|$, the line $\text{Im } \zeta = \text{Im } w_n$ is, for example, above the crosses, and we can not take a contour like Γ in this figure. However, in this case we have only to deform Γ and take a contour topologically equivalent to Γ' .

$J_{BH} \neq 0$ case. First, from (3.17) and (3.18), we get

$$\begin{aligned} \frac{\partial}{\partial \mu} S(\beta) \Big|_{\beta=\beta_H} &= -\frac{i}{4l^2} \Omega_H \beta_H^2 \sum_{n=-\infty}^{\infty} \int_{r_+^2}^{\infty} d(r^2) \lim_{r \rightarrow r'} \left[G_F^E(z_n; \beta_H) \right. \\ &\quad \left. - B \int_{A_n+B}^{\infty} dz G_F^E(z; \beta_H) \frac{1}{\sqrt{(z-A_n)^2 - B^2}} \frac{c_n(z-A_n) + B}{(z-A_n + c_n B)^2} \right] \Big|_{\Delta\tau=\Delta\varphi_E=0}. \end{aligned}$$

The first term is nothing but $\partial_{\mu}(\beta F(\beta)|_{\beta=\beta_H})$. By integrating the above expression, we get the exact expression of the entropy

$$\begin{aligned} S(\beta_H) &= \beta_H F(\beta_H) + \frac{\Omega_H}{8\pi l^3} \beta_H^2 \sum_{n=-\infty}^{\infty} \int_{r_+^2}^{\infty} d(r^2) \lim_{r \rightarrow r'} \left[B \int_{A_n+B}^{\infty} dz \right. \\ &\quad \left. \times \frac{X^{1-\lambda} \{1 + (\lambda-1) \ln X\}}{\ln^2 X \sqrt{z^2 - 1}} \frac{1}{\sqrt{(z-A_n)^2 - B^2}} \frac{c_n(z-A_n) + B}{(z-A_n + c_n B)^2} \right] \Big|_{\Delta\tau=\Delta\varphi_E=0} + c. \end{aligned} \quad (3.19)$$

Here

$$X = z + \sqrt{z^2 - 1},$$

and c is a constant independent of μ , which is dropped in the flat case.

For $J_{BH} = 0$, we similarly get

$$\begin{aligned} S(\beta_H) &= \frac{1}{4l^3} \beta_H \sum_{n=-\infty}^{\infty} \int_{r_+^2}^{\infty} d(r^2) \lim_{r \rightarrow r'} \left[B \int_{A_n+B}^{\infty} dz \right. \\ &\quad \left. \times \frac{X^{1-\lambda} \{1 + (\lambda-1) \ln X\}}{\ln^2 X \sqrt{z^2 - 1}} \frac{1}{\sqrt{(z-A_n)^2 - B^2}} \frac{c_n(z-A_n) - B}{(z-A_n + c_n B)^2} \right] \Big|_{\Delta\tau=\Delta\varphi_E=0} + c. \end{aligned} \quad (3.20)$$

This can be obtained also from the second term in (3.19) by the replacement $\Omega_H \beta_H \rightarrow 2\pi$. The term corresponding to the first term in (3.19) is absent because the power of β in $\text{Tr } G_{BH}^E$ is different.

As we have exact expression of the entropy, we can study its properties without any ambiguity [35]. Here, we will concentrate on the structure of the divergences coming from short-distance behavior. In the recent arguments [37]-[40], these divergences are considered to be important for understanding the black hole entropy and quantum aspects of gravity. We find that the short-distance divergences come from (i) the contribution from taking the trace (ordinary ultraviolet divergences in statistical field theory), or (ii) the integration near the outer horizon.

To see this, we introduce a variable ρ defined by $\rho^2 = r^2 - r_+^2$, and cutoffs

$$s^2 = r'^2 - r^2$$

for the trace, and

$$r_+^2 \rightarrow r_+^2 + \epsilon_H^2$$

for the integration near the horizon. In the limit $\rho, s \rightarrow 0$, we have

$$\begin{aligned} A_n &\sim c_n^+ \left(1 + \frac{1}{d_H^2} (\rho^2 + s^2/2) \right), \quad B = \frac{\rho}{d_H^2} \sqrt{\rho^2 + s^2}, \\ z_{n=0}^0 &= z(x, x') \Big|_{\Delta\tau = \Delta\varphi_E = 0} \sim 1 + \frac{s^4}{8(r^2 - r_+^2)(r^2 - r_-^2)} \quad \text{for } \rho^2 \gg s^2 \\ &\sim 1 + \frac{1}{d_H^2} \left(\rho^2 + \frac{s^2}{2} - \rho \sqrt{\rho^2 + s^2} \right) \quad \text{otherwise.} \end{aligned}$$

Then we find that the ultraviolet divergences come only from the term with $n = 0$ in (3.19) and (3.20) (including the $n = 0$ term of $F(\beta_H)$ in (3.19)).

The integrand in the $n = 0$ term of $F(\beta_H)$ becomes divergent for small s like

$$\frac{1}{\sqrt{\sigma(x, x')}} \sim \frac{1}{s^2} \sqrt{(r^2 - r_+^2)(r^2 - r_-^2)}.$$

This can be regarded as an ordinary ultraviolet divergence of field theory since $\sqrt{\sigma(x, x')}$ is a distance.

The term relevant to the other divergences for $J_{BH} \neq 0$ is the $n = 0$ term in the summation in (3.19). We rewrite this term by $z' \equiv z - A_0$ and $\delta = A_0 - 1 \sim d_H^{-2} (\rho^2 + s^2/2)$. Noting that $\ln X \sim \sqrt{2(z' + \delta)}$ up to $\mathcal{O}(z', \delta)$, we find the expression of the divergences;

$$I \sim \frac{\Omega_H}{8\pi l^3} \beta_H^2 \int_{\epsilon_H^2} d(\rho^2) B \int_B dz' \frac{1 + (\lambda - 1)\sqrt{2(z' + \delta)}}{(z' + \delta)(z' + B)\sqrt{z' + \delta}\sqrt{z'^2 - B^2}}$$

$$\begin{aligned}
&= \frac{\Omega_H}{8\pi l^3} \beta_H^2 \int_{\epsilon_H^2} d(\rho^2) B \int^1 dv \\
&\quad \times \frac{1}{(1 + (\delta/B)v)(1 + v)\sqrt{1 - v^2}} \left\{ \frac{1}{\sqrt{1 + (\delta/B)v}} \left(\frac{v}{B} \right)^{3/2} + \sqrt{2}(\lambda - 1) \frac{v}{B} \right\},
\end{aligned}$$

where $v = B/z'$. Here we have two cases. First, for $\epsilon_H, \rho \simeq s$ or $\epsilon_H, \rho \gg s$, we obtain

$$\begin{aligned}
I &\sim \Omega_H \beta_H^2 l^{-3} \int_{\epsilon_H^2} d(\rho^2) (B^{-3/2} + c B^{-1}) \\
&\sim \Omega_H \beta_H^2 d_H^2 l^{-3} (d_H/\epsilon_H + c' \ln(d_H/\epsilon_H)),
\end{aligned}$$

where c and c' are constants. On the other hand, for $\epsilon_H, \rho \ll s$, we obtain

$$\begin{aligned}
I &\sim \Omega_H \beta_H^2 l^{-3} \int_{s^2} d(\rho^2) (\delta^{-3/2} + c \delta^{-1}) \\
&\sim \Omega_H \beta_H^2 d_H^2 l^{-3} (d_H/s + c' \ln(d_H/s)).
\end{aligned}$$

Therefore the divergences are given in terms of the larger cutoff, i.e., $\max\{\epsilon_H, s\}$. Namely, we have only to introduce s or ϵ_H in order to regulate the divergences.

Similarly, the divergences for $J_{BH} = 0$ is obtained from the above by the replacement $\Omega_H \beta_H \rightarrow 2\pi$.

From the expression of the entropy, we find various divergences coming from short distance such as ϵ_H^{-1} , $\ln \epsilon_H^2$, and s^{-2} . However, all of them are not due to the existence of the outer horizon. The expression of the divergences depends largely upon the regularization schemes. If we adopt the regularization related to the horizon, they are expressed by the cutoff related to the horizon. But if not, this is not the case. In addition, the leading divergent term is proportional to $\Omega_H d_H^3 \beta_H^2 l^{-3} = r_+ r_- / d_H$ for $J_{BH} \neq 0$ and to $d_H^3 \beta_H l^{-3} = r_+^2 / l$ for $J_{BH} = 0$. Therefore, they are not proportional to the area of the horizon.

These results might be due to the special properties of the BTZ black hole. But they seem to indicate the importance of regularization schemes and of curvature effects in a truly curved spacetime. We should carefully examine what regularization scheme is physical so that we justify the claim in the literature that the horizon plays an important role in black hole thermodynamics.

3.5 Discussion

In this chapter, we explored the thermodynamics of scalar fields in the BTZ black hole background in the framework of quantum field theory in curved spacetime. We took two approaches. One was based on mode expansion of the scalar field and summation over states. We obtained mode functions and explicit forms of densities of states, free energies and entropies. We did not need the WKB approximation. We found that the

thermodynamic quantities depended largely upon boundary conditions. In particular, divergent terms of the entropy were not necessarily due to the existence of the outer horizon. In addition, the partition function did not take the form (3.3), and the entropy was not proportional to the area of the outer horizon.

In the other approach, we used Hartle-Hawking Green functions. For scalar fields with a generic mass squared, we obtained the Hartle-Hawking Green functions and Green functions on a cone geometry. Moreover, we obtained exact expressions of free energies and entropies. We did not need the heat kernel expansion. We again found that the free energy did not take the form (3.3) and the entropy was not proportional to the area of the horizon. The divergences of the entropy were not necessarily due to the horizon. They depended upon regularization schemes. The thermodynamic quantities were different from those in the previous approach.

We can imagine several reasons that our results do not agree with those in the literature: (i) Three dimensional spacetime and four dimensional spacetime (and D -dimensional spacetime considered in [37]-[40]) are different; in the former, gravitons do not exist and the meaning of gravitational coupling is not clear. (ii) The BTZ black hole is different from a four dimensional black hole; it is not asymptotically flat, but it approaches AdS_3 , and it has a cosmological constant (fundamental scale). (iii) The results in [37]-[40] are obtained in the Rindler (flat) limit, and they may receive large corrections by curvature effects.

It is worthwhile investigating that reason further. However, we should note that our results indicate the importance of curvature effects and of precise discussions on boundary conditions, regularization schemes, and relations among methods of calculation. We saw that the claims in the literature were largely affected by them.⁹ Thus without settling these problems we cannot understand the meaning of the entropy and its relation to the information paradox and the renormalization of the gravitational coupling constant.

At present, we have no “definition” of thermodynamics of a quantum field in a black hole background. We have to look for it taking into account physical consistency as a clue (if black hole thermodynamics is truly sensible). We have to determine the “correct” boundary condition and prescription to calculate the thermodynamic quantities. Admittedly, this is not an easy task. In the case of the BTZ black hole, one possibility is to utilize the result of \widetilde{AdS}_3 . Quantization of a scalar field in \widetilde{AdS}_3 has been well studied [49]-[51]. One knows the inner product and the complete basis. As discussed in the next chapter, \widetilde{AdS}_3 is the same as $\widetilde{SL}(2, R)$ and the mode functions are expressed by the matrix

⁹Recently, entropies of a scalar field in a BTZ black hole background were discussed in the WKB approximation [57] and in the heat kernel expansion [58]. In the former approach, the entropy is proportional to the area. In the latter, they argued that the divergences can be absorbed into the renormalization of the gravitational coupling constant and the cosmological constant. But the entropy is not proportional to the area. The relation among them and ours are not clear.

elements of $\widetilde{SL}(2, R)$ representations in the elliptic basis (see appendix C). On the other hand, our mode functions in (3.7) are the matrix elements in the hyperbolic basis. Thus we may obtain a sensible quantum theory from the one in \widetilde{AdS}_3 by changing the basis of $\widetilde{SL}(2, R)$ representations. In turn this may settle the problems. Another possibility is to investigate the string theory in the BTZ black hole background because string theory is regarded as the fundamental theory including gravity. This is the subject in the next chapter.

In the study of quantum black holes, it has been hardly possible to make precise discussions and to examine various claims. This is because we have no quantum theory of gravity and a system of quantum fields plus a black hole is quite complicated. In our model, we could discuss the thermodynamics of a quantum scalar field in the BTZ background without ambiguity. Therefore we believe that our results in this chapter may provide reliable bases for further investigations of this subject.

4 STRING THEORY IN THE THREE DIMENSIONAL BLACK HOLE GEOMETRY

Now we begin a new approach to the three dimensional quantum black holes. In this chapter, we will discuss the string theory in this black hole geometry in the framework of conformal field theory [41]. One of the motivations is to consider the open problems in the previous chapter and understand the microscopic origin of black hole entropy. However, our main purpose here is more general and it is to pursue fundamental problems of quantum gravity as discussed in the introduction. Since the string theory is regarded as the fundamental theory including gravity, we expect to get a deeper understanding of quantum aspects of the three dimensional black hole. From the string-theory point of view, this work also has a significance as the first attempt to quantize a string in a black hole background with an infinite number of propagating modes. Moreover, the analysis here serves as an investigation of a string theory in a non-trivial background.

After the BTZ black hole was found, it was soon realized that a slight modification of the solution yields a solution (an exact background) of the bosonic string theory [33, 34]. This is one of the few known exact solutions in string theory and one of the simplest solutions. However, any detailed analyses of this string theory had not been made. In addition, a string in a curved spacetime has not been well understood and it is not clear whether a string in a black hole background can be physically sensible. Therefore we will make detailed analyses and study consistency conditions of the string in the three dimensional black hole geometry [41]. It turns out that we can investigate this model in an explicit manner owing to its simplicity. We analyze the spectrum by solving the level matching condition and obtain winding modes. We then study the ghost problem

and show explicit examples of physical states with negative norms. We discuss general properties of the tachyon propagation and the target-space geometry which are irrelevant to the details of the spectrum. We find a self-dual T-duality transformation reversing the black hole mass. The existence of the ghosts indicates that our model is not physical as it is. Thus we also discuss possibilities to obtain a sensible string theory.

4.1 The three dimensional black hole as a string background

First, we review the three dimensional black hole from the string-theory point of view. In the context of string theory, the three dimensional black hole is described by an orbifold of the $\widetilde{SL}(2, R)$ WZW model [33, 34].

4.1.1 Description using the $\widetilde{SL}(2, R)$ WZW model

In string theory, we start from the $SL(2, R)$ WZW model ¹⁰ with action

$$\frac{k}{8\pi} \int_{\Sigma} d^2\sigma \sqrt{h} h^{\alpha\beta} \text{Tr} \left(g^{-1} \partial_{\alpha} g g^{-1} \partial_{\beta} g \right) + ik\Gamma(g),$$

where $h_{\alpha\beta}$ is the metric of a Riemann surface Σ , g is an element of $SL(2, R)$ and k is the level of the WZW model. Γ is the Wess-Zumino term given by

$$\frac{1}{12\pi} \int_{B_{\Sigma}} \text{Tr} \left(g^{-1} dg \wedge g^{-1} dg \wedge g^{-1} dg \right),$$

where B_{Σ} is a three manifold with boundary Σ . We parametrize g by

$$g = \begin{pmatrix} X_1 + X_2 & X_3 + X_0 \\ X_3 - X_0 & X_1 - X_2 \end{pmatrix},$$

$$\det g = X_0^2 + X_1^2 - X_2^2 - X_3^2 = 1.$$

The latter equation is nothing but the embedding equation of AdS_3 in a flat space. ¹¹ Thus $SL(2, R)$ and AdS_3 are the same manifold. This is the essence of the reason why the BTZ black hole is described by using the $SL(2, R)$ WZW model. By setting $X_i = x_i/l$, we obtain the direct correspondence to chapter 2.

As before, in order to decompactify the time direction of $SL(2, R)$, we go to the universal covering group $\widetilde{SL}(2, R)$, and consider three regions parametrized by (2.3). Furthermore, we make the change of variables (2.4), identify φ with $\varphi + 2\pi$, and cut out the region $r^2 < 0$. Consequently, the WZW action takes the form

$$S = \frac{1}{4\pi\alpha'} \int d^2\sigma \sqrt{h} \left(h^{\alpha\beta} G_{\mu\nu} + i\epsilon^{\alpha\beta} B_{\mu\nu} \right) \partial_{\alpha} X^{\mu} \partial_{\beta} X^{\nu},$$

¹⁰There are difficulties to construct a CFT based on a non-compact group manifold. In this thesis, we will simply assume the existence of the $\widetilde{SL}(2, R)$ WZW model.

¹¹Note that X_i are dimensionless.

where $(2\pi\alpha')^{-1}$ is the string tension. $G_{\mu\nu}$ and $B_{\mu\nu}$ are given by

$$\begin{aligned} ds_{\text{string}}^2 &= G_{\mu\nu} dX^\mu dX^\nu = \frac{\alpha' k}{l^2} ds_{BH}^2 \\ B &= \frac{\alpha' k}{l^2} r^2 d\varphi \wedge d(t/l). \end{aligned} \quad (4.1)$$

B is defined up to exact forms. Comparing the above metric with $g_{\mu\nu}^{BH}$ in (2.5), we find that $G_{\mu\nu}$ represents the three dimensional black hole with $l^2 = \alpha' k$.

Since the model is described by a WZW model, the background geometry maintains the conformal invariance of the world sheet to all orders in α' ; the geometry gives an exact background (a solution to string theory). The exact geometry, which is read off from the full quantum effective action, is given simply by the replacement $k \rightarrow k - 2$ [59]. Here -2 is the second Casimir of the adjoint representation of $sl(2, R)$. Then one has

$$l^2 = (k - 2)\alpha'. \quad (4.2)$$

One can confirm that the above geometry is a solution to the low energy effective theory [33]. In three dimensions, the low energy string action is

$$S_{LEET} = \int d^3x \sqrt{-G} e^{-2\phi} \left[\frac{2(26 - D)}{3\alpha'} + R + 4(\nabla\phi)^2 - \frac{1}{12} H_{\mu\nu\rho} H^{\mu\nu\rho} \right],$$

where ϕ is the dilaton, $H = dB$, and $D = 3$. The equations of motion derived from this action are

$$\begin{aligned} R_{\mu\nu} + 2\nabla_\mu \nabla_\nu \phi - \frac{1}{4} H_{\mu\rho\sigma} H_\nu^{\rho\sigma} &= 0 \\ \nabla^\mu (e^{-2\phi} H_{\mu\nu\rho}) &= 0 \\ 4\nabla^2 \phi - 4(\nabla\phi)^2 + \frac{2(26 - D)}{3\alpha'} + R - \frac{1}{12} H^2 &= 0. \end{aligned}$$

A special property of three dimensions is that $H_{\mu\nu\rho}$ must be proportional to the volume form $\epsilon_{\mu\nu\rho}$. Then, by setting $\phi = 0$, the second equation yields $H_{\mu\nu\rho} = (2/l)\epsilon_{\mu\nu\rho}$. Substituting this into the first equation gives

$$R_{\mu\nu} = -\frac{2}{l^2} G_{\mu\nu}.$$

This is exactly the Einstein's equation with a negative cosmological constant $-l^{-2}$. The third equation is satisfied if

$$l^2 = \frac{6\alpha'}{23}. \quad (4.3)$$

Therefore every solution to three dimensional general relativity with negative cosmological constant is a solution to low energy string theory with $\phi = 0$, $H_{\mu\nu\rho} = (2/l)\epsilon_{\mu\nu\rho}$, and $l^2 = 6\alpha'/23$. In particular, the geometry (4.1) is a solution with

$$B_{\varphi t} = \frac{r^2}{l}, \quad \phi = 0.$$

The level k is determined by (4.2) and (4.3):

$$k = \frac{52}{23}. \quad (4.4)$$

In the following, we will set $l = 1$ for brevity. It can be recovered simply by counting of dimension.

4.1.2 Chiral currents and the stress tensor

Next, let us summarize the properties of currents for later use. The $\widetilde{SL}(2, R)$ WZW model has a chiral $\widetilde{SL}(2, R)_L \times \widetilde{SL}(2, R)_R$ symmetry. The currents associated with this symmetry are given by

$$J(z) = \frac{ik}{2} \partial g g^{-1}, \quad \tilde{J}(\bar{z}) = \frac{ik}{2} g^{-1} \bar{\partial} g, \quad (4.5)$$

where $z = e^{\tau+i\sigma}$ and $\bar{z} = e^{\tau-i\sigma}$. The currents act on g as

$$J^a(z)g(w, \bar{w}) \sim \frac{-\tau^a g}{z-w}, \quad \tilde{J}^a(\bar{z})g(w, \bar{w}) \sim \frac{-g \tau^a}{\bar{z}-\bar{w}}. \quad (4.6)$$

Here, we have defined J^a ($a = 0, 1, 2$) by $J(z) = \eta_{ab}\tau^a J^b(z)$ and similarly for \tilde{J}^a , where $\eta_{ab} = \text{diag}(-1, 1, 1)$. τ^a form a basis of $sl(2, R)$ with the properties

$$[\tau^a, \tau^b] = i\epsilon^{ab}_c \tau^c, \quad \text{Tr}(\tau^a \tau^b) = -\frac{1}{2} \eta^{ab}.$$

In terms of the Pauli matrices, $\tau^0 = -\sigma^2/2$, $\tau^1 = i\sigma^1/2$ and $\tau^2 = i\sigma^3/2$. From the currents, one obtains the stress tensor

$$T(z) = \frac{1}{k-2} \eta_{ab} J^a(z) J^b(z).$$

Then the conformal modes of the currents and the stress tensor satisfy the commutation relations

$$\begin{aligned} [J_n^a, J_m^b] &= i\epsilon^{ab}_c J_{n+m}^c + \frac{k}{2} n \eta^{ab} \delta_{m+n}, \\ [L_n, J_m^a] &= -m J_{n+m}^a, \\ [L_n, L_m] &= (n-m)L_{n+m} + \frac{c}{12} n(n^2-1) \delta_{n+m}, \end{aligned}$$

where $c = 3k/(k-2)$. For the critical value $c = 26$, we have $k = 52/23$. This is consistent with (4.4). The above Kac-Moody algebra is expressed in the basis $I_n^\pm \equiv J_n^1 \pm iJ_n^2$ and $I_n^0 \equiv J_n^0$ as

$$\begin{aligned} [I_n^+, I_m^-] &= -2I_{n+m}^0 + kn\delta_{n+m}, & [I_n^\pm, I_m^\pm] &= 0, \\ [I_n^0, I_m^\pm] &= \pm I_{n+m}^\pm, & [I_n^0, I_m^0] &= -\frac{k}{2}n\delta_{n+m}. \end{aligned}$$

On the other hand, in the basis $J_n^\pm \equiv J_n^0 \pm J_n^1$ and J_n^2 , the algebra is written as

$$\begin{aligned} [J_n^+, J_m^-] &= -2iJ_{n+m}^2 - kn\delta_{n+m}, & [J_n^\pm, J_m^\pm] &= 0, \\ [J_n^2, J_m^\pm] &= \pm iJ_{n+m}^\pm, & [J_n^2, J_m^2] &= \frac{k}{2}n\delta_{n+m}. \end{aligned} \quad (4.7)$$

Note that the Hermitian conjugates for the latter basis are given by

$$(J_m^\pm)^\dagger = J_{-m}^\pm, \quad (J_m^2)^\dagger = J_{-m}^2. \quad (4.8)$$

Similar expressions hold for the anti-holomorphic part.

4.1.3 Twisting

As explained before, in order to get the three dimensional black hole, we have (i) to go to the universal covering space of $SL(2, R)$, (ii) to make the identification $\varphi \sim \varphi + 2\pi$ and (iii) to drop the region $r^2 < 0$. We can take (i) into account by considering the representation theory of $\widetilde{SL}(2, R)$ instead of $SL(2, R)$. The point (iii) was related to the problem of closed timelike curves (see chapter 2 and [31, 33]); we will discuss this point in section 4.5. In terms of string theory, (ii) represents a twist of $\widetilde{SL}(2, R)$, and we will concentrate on (ii) for now.

In order to express the identification in (ii) by the $sl(2, R)$ currents, it is convenient to parametrize the group manifold by analogs of Euler angles; we parametrize Region I-III by [41, 16]

$$\begin{aligned} \text{Region I} : g &= e^{-i\theta_L\tau^2} e^{-i\rho\tau^1} e^{-i\theta_R\tau^2} &= \begin{pmatrix} e^\varphi \cosh \rho/2 & e^{\hat{t}} \sinh \rho/2 \\ e^{-\hat{t}} \sinh \rho/2 & e^{-\varphi} \cosh \rho/2 \end{pmatrix}, \\ \text{Region II} : g &= e^{-i\theta_L\tau^2} e^{-i\rho\tau^0} e^{-i\theta_R\tau^2} &= \begin{pmatrix} e^\varphi \cos \rho/2 & e^{\hat{t}} \sin \rho/2 \\ -e^{-\hat{t}} \sin \rho/2 & e^{-\varphi} \cos \rho/2 \end{pmatrix}, \\ \text{Region III} : g &= e^{-i\theta_L\tau^2} s e^{-i\rho\tau^1} e^{-i\theta_R\tau^2} &= \begin{pmatrix} e^\varphi \sinh \rho/2 & e^{\hat{t}} \cosh \rho/2 \\ -e^{-\hat{t}} \cosh \rho/2 & -e^{-\varphi} \sinh \rho/2 \end{pmatrix}, \end{aligned} \quad (4.9)$$

where $s = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$,

$$\theta_L = \hat{\varphi} + \hat{t}, \quad \theta_R = \hat{\varphi} - \hat{t}, \quad (4.10)$$

and

$$\begin{aligned} \text{Region I} &: \hat{r} = \cosh \rho/2, & \sqrt{\hat{r}^2 - 1} &= \sinh \rho/2, \quad (\rho > 0), \\ \text{Region II} &: \hat{r} = \cos \rho/2, & \sqrt{1 - \hat{r}^2} &= \sin \rho/2, \quad (\pi > \rho > 0), \\ \text{Region III} &: \sqrt{-\hat{r}^2} = \sinh \rho/2, & \sqrt{1 - \hat{r}^2} &= \cosh \rho/2, \quad (\rho > 0). \end{aligned}$$

The currents (4.5) then take the form, e.g.,

$$J^2 = \frac{k}{2} \left(\partial \theta_L + (2\hat{r}^2 - 1) \partial \theta_R \right), \quad \tilde{J}^2 = \frac{k}{2} \left(\bar{\partial} \theta_R + (2\hat{r}^2 - 1) \bar{\partial} \theta_L \right). \quad (4.11)$$

From (2.4), we find that the translation of φ is given by a linear combination of those of \hat{t} and $\hat{\varphi}$. From the AdS_3 point of view, the translations of \hat{t} and $\hat{\varphi}$ corresponded to boosts in the flat spacetime in which AdS_3 was embedded. On the other hand, in the context of the $SL(2, R)$ WZW model, the translations of \hat{t} and $\hat{\varphi}$ correspond to a vector and an axial symmetry generated by $J_0^2 \pm \tilde{J}_0^2$ [33] (see (4.6)). Therefore the translation of φ is generated by $Q_\varphi \equiv \Delta_- J_0^2 + \Delta_+ \tilde{J}_0^2$, where

$$\Delta_\pm = r_+ \pm r_-.$$

Then $\delta\varphi = 2\pi$ with fixed t is expressed by

$$\Delta_+ \delta\theta_L = \Delta_- \delta\theta_R = 2\pi \Delta_+ \Delta_- . \quad (4.12)$$

For describing the black hole, we have to twist (orbifold) the WZW model with respect to this discrete group. In the following, we denote it by Z_φ and hence we will call our black hole the $\widetilde{SL}(2, R)/Z_\varphi$ black hole.

Note that, if one gauges the vector or the axial symmetry, the resulting coset theory describes the $SL(2, R)/U(1)$ black hole [11].

4.2 The spectrum of a string on $\widetilde{SL}(2, R)/Z_\varphi$ orbifold

As a consequence of the identification $\varphi \sim \varphi + 2\pi$, twisted (winding) sectors arise in the theory. In this section, we will discuss the spectrum including the twisted (winding) sectors [41]. One difficulty here is that the field φ is not a free field. We are working in a group manifold, so we cannot use the argument for flat theories. However, a similar orbifolding has been discussed in [60] to construct a $SU(2)/Z_N$ orbifold. Thus we will follow that argument and solve the level matching condition; this is required from various kinds of consistency of string theory, for example, modular invariance and the invariance under the shift of the world-sheet spatial coordinate. Other consistency conditions such as unitarity should also be checked. We will discuss them in section 4.3 and 4.5. These consistency conditions are closely related to each other.

4.2.1 Kac-Moody Primaries in the $\widetilde{SL}(2, R)$ WZW model

Before discussing the orbifolding, let us consider Kac-Moody primaries in the $\widetilde{SL}(2, R)$ WZW model. Operators are Kac-Moody primary if they form irreducible representations of global $\widetilde{SL}(2, R)_L \times \widetilde{SL}(2, R)_R$ and if they are annihilated by the Kac-Moody generators J_n^a and \tilde{J}_n^a for $n > 0$. For WZW models, they are also Virasoro primary. For a unitary theory based on a compact group, local fields (wave functions) on the group correspond to Kac-Moody primaries and they are given by the matrix elements of the unitary representations of the group [60, 61]. We are also interested in a unitary string theory. Thus, we start from the Kac-Moody primary fields which correspond to the matrix elements of the unitary representations of $\widetilde{SL}(2, R)$. They have local expressions in θ_L, θ_R and ρ without derivatives of these fields,

$$V(\theta_L(z, \bar{z}), \theta_R(z, \bar{z}), \rho(z, \bar{z})) .$$

For $\widetilde{SL}(2, R)$, we have five types of unitary representations [62], namely, the identity representation, the principal continuous series, the complementary series, the highest and the lowest discrete series. In order to express the matrix elements of these representations, we have to further specify the basis of the representation. In representations of $\widetilde{SL}(2, R)$, one has three types of basis. Let us denote the generators of $sl(2, R)$ by J^0, J^1 and J^2 . Then, the bases diagonalizing J^0, J^2 and $J^0 - J^1$ are called elliptic, hyperbolic and parabolic, respectively. Since we are interested in the orbifolding related to the action of J_0^2 and \tilde{J}_0^2 , we consider representations in the hyperbolic basis. This basis has been used in the study of the Minkowskian $SL(2, R)/U(1)$ black hole [16, 17]. Consequently, the Kac-Moody primaries other than the identity are expressed by the matrix elements as

$$\begin{aligned} {}^{P(C)}D_{J, J'}^\chi(g) & \quad \text{for the principal continuous (P) and the complementary (C) series,} \\ {}^{H(L)}D_{J, J'}^j(g) & \quad \text{for the highest (H) and the lowest (L) weight series,} \end{aligned} \quad (4.13)$$

where j labels the value of the Casimir; J and J' refer to the eigenvalue of J^2 . For the principal continuous and the complementary series, one has additional parameters, $0 \leq m_0 < 1$ specifying the representation, and \pm specifying the base state. χ is the pair (j, m_0) . Under this construction, the primary fields have the common j -value in the left and the right sector. Note that the spectrum of J^2 ranges all over the real number, i.e., $J, J' \in \mathbf{R}$. For the details, see appendix C.

Here some remarks may be in order. First, one can explicitly construct the primary fields belonging to the unitary representations of $\widetilde{SL}(2, R)$ using free field realizations of the $sl(2, R)$ Kac-Moody algebra [26]. Second, in section 4.4, we will find the correspondence between the above primary fields and the Klein-Gordon fields discussed in chapter 3. Third, suppose [16] that the Kac-Moody primary fields lead to normalizable operators,

and that the CFT inherits the natural inner product of the $\widetilde{SL}(2, R)$ representations. Then the Kac-Moody primaries should be given by the matrix elements of the unitary representations (except for the complementary series) because a complete basis for the square integrable functions on $\widetilde{SL}(2, R)$ is given by them. Fourth, most of our discussions below do not change even if we start from other representations at the base. We easily find that the theory becomes non-unitary if we start from non-unitary ones as in [17]. Finally, we cannot deny the possibility of the Kac-Moody primaries which are non-local and/or contain derivatives of the coordinate fields. However, in our understanding, such a possibility has not been found so far.

4.2.2 Vertex operators in the $\widetilde{SL}(2, R)/Z_\varphi$ theory

We now turn to the $\widetilde{SL}(2, R)/Z_\varphi$ theory and consider the vertex operators [41]. First, we construct the operator which expresses the twisting. Let us recall that the chiral currents $J^2(z)$ and $\tilde{J}^2(\bar{z})$ have the operator product expansions (OPE)

$$J^2(z)J^2(0) \sim \frac{k/2}{z^2}, \quad \tilde{J}^2(\bar{z})\tilde{J}^2(0) \sim \frac{k/2}{\bar{z}^2}.$$

So, we represent them by free fields $\theta_L^F(z)$ and $\theta_R^F(\bar{z})$ as

$$J^2(z) = \frac{k}{2}\partial\theta_L^F, \quad \tilde{J}^2(\bar{z}) = \frac{k}{2}\bar{\partial}\theta_R^F.$$

The normalization of the fields is fixed by

$$\theta_L^F(z)\theta_L^F(0) \sim +\frac{2}{k}\ln z, \quad \theta_R^F(\bar{z})\theta_R^F(0) \sim +\frac{2}{k}\ln \bar{z}.$$

The signs are opposite to the usual case because of the negative metric of the J^2 direction. The explicit forms of θ_L^F and θ_R^F are obtained by integration of (4.11). The local integrability is assured by the current conservation. In addition, we introduce $\theta_L^{NF}(z, \bar{z})$ and $\theta_R^{NF}(z, \bar{z})$ by

$$\theta_L(z, \bar{z}) = \theta_L^F(z) + \theta_L^{NF}(z, \bar{z}), \quad \theta_R(z, \bar{z}) = \theta_R^F(\bar{z}) + \theta_R^{NF}(z, \bar{z}).$$

Note θ_L^{NF} and θ_R^{NF} are not free fields. Then, the twisting operator with winding number $n_w \in \mathbf{Z}$ is given by

$$W(z, \bar{z}; n_w) \equiv \exp \left\{ -i\frac{k}{2}n_w (\Delta_- \theta_L^F - \Delta_+ \theta_R^F) \right\}.$$

Indeed, this has the OPE's

$$\begin{aligned} \theta_L^F(z)W(0, \bar{z}; n_w) &\sim -in_w \Delta_- \ln z \cdot W(0, \bar{z}; n_w), \\ \theta_R^F(\bar{z})W(z, 0; n_w) &\sim +in_w \Delta_+ \ln \bar{z} \cdot W(z, 0; n_w). \end{aligned}$$

Thus, θ_L^F and θ_R^F shift by $2\pi\Delta_-n$ and $2\pi\Delta_+n$, respectively, under the translation of the world-sheet coordinate $\sigma \rightarrow \sigma + 2\pi$, i.e., $z \rightarrow e^{2\pi i}z$ and $\bar{z} \rightarrow e^{-2\pi i}\bar{z}$. Namely, $\delta\varphi = 2\pi n_W$ and $\delta t = 0$ on $W(z, \bar{z}; n_W)$ under $\delta\sigma = 2\pi$. Hence, $W(z, \bar{z}; n_W)$ expresses the correct twisting.

Then we readily obtain the primary fields in our model. First, a general untwisted primary field takes the form (4.13). In our parametrization (4.9), it is given by

$$V_{J_L, J_R}^j(z, \bar{z}; 0) = D_{J_L, J_R}^j(g'(\rho)) e^{-iJ_L\theta_L - iJ_R\theta_R}, \quad (4.14)$$

where we have omitted irrelevant indices of the matrix elements. The explicit form of $g'(\rho)$ depends on which region we consider. Second, combining the untwisted primary field and the twisting operator W , we obtain a general primary field in the $\widetilde{SL}(2, R)/Z_\varphi$ black hole CFT [41]:

$$\begin{aligned} V_{J_L, J_R}^j(z, \bar{z}; n_W) &= V_{J_L, J_R}^j(z, \bar{z}; 0)W(z, \bar{z}; n_W), \\ &= D_{J_L, J_R}^j(g'(\rho)) \exp \left\{ -i \left(J'_L\theta_L^F + J_L\theta_L^{NF} + J'_R\theta_R^F + J_R\theta_R^{NF} \right) \right\}, \end{aligned} \quad (4.15)$$

where

$$J'_L = J_L + \frac{k}{2}\Delta_-n_W, \quad J'_R = J_R - \frac{k}{2}\Delta_+n_W.$$

From this primary field, we find that a general vertex operator takes the form

$$J_N \cdot \tilde{J}_{\tilde{N}} \cdot V_{J_L, J_R}^j(z, \bar{z}; n_W),$$

where J_N and $\tilde{J}_{\tilde{N}}$ stand for generic products of the Kac-Moody generators J_{-n}^a and \tilde{J}_{-n}^a , respectively. Here we have a restriction on the above form because of the orbifolding. Note that the untwisted part depends on θ_L^F and θ_R^F as $\exp(-i\omega_L\theta_L^F - i\omega_R\theta_R^F)$ and the full operator as $\exp(-i\omega'_L\theta_L^F - i\omega'_R\theta_R^F)$, where

$$\omega_L^{(\prime)} = J_L^{(\prime)} + i(N_+ - N_-), \quad \omega_R^{(\prime)} = J_R^{(\prime)} + i(\tilde{N}_+ - \tilde{N}_-);$$

N_\pm and \tilde{N}_\pm are the number of J_{-n}^\pm and \tilde{J}_{-n}^\pm , respectively. This follows from the fact that $J_{-n}^\pm(\tilde{J}_{-n}^\pm)$ shifts $\omega_L(\omega_R)$ by $\pm i$ because of the commutation relation (4.7).¹² The vertex operator cannot be single-valued on $\widetilde{SL}(2, R)/Z_\varphi$ orbifold if $\omega_{L,R}^{(\prime)}$ are complex. Thus $N_+ = N_-$ and $\tilde{N}_+ = \tilde{N}_-$, namely, $\omega_{L,R}^{(\prime)} = J_{L,R}^{(\prime)}$ should hold. Consequently, the vertex operators in our model are given by

$$K_{-n}^a \cdots \tilde{K}_{-m}^b \cdots V_{J_L, J_R}^j(z, \bar{z}; n_W), \quad (4.16)$$

where K_{-n}^a and \tilde{K}_{-n}^a ($a = +, -, 2$) are defined by

$$K_{-n}^+ = J_{-n}^+ J_0^-, \quad K_{-n}^- = J_{-n}^- J_0^+, \quad K_{-n}^2 = J_{-n}^2,$$

and similar expressions for \tilde{K}_{-n}^a .

¹²This seems to contradict the Hermiticity of $J_0^2(\tilde{J}_0^2)$. However, this is not the case because the spectrum of $J_0^2(\tilde{J}_0^2)$ is continuous. Representations of $SL(2, R)$ in the hyperbolic basis are described in appendix C.

4.2.3 Level matching

Now we consider the level matching condition to further discuss the spectrum. To obtain the expressions of L_0 and \tilde{L}_0 for the vertex operators, we first decompose the stress tensor following GKO (Goddard-Kent-Olive) [63]. For the holomorphic part, we then have

$$\begin{aligned} T(z) &= T^{sl(2,R)/so(1,1)}(\rho, \theta_L^{NF}, \theta_R^{NF}) + T^{so(1,1)}(\theta_L^F), \\ T^{so(1,1)}(\theta_L^F) &= +\frac{k}{4}\partial\theta_L^F\partial\theta_L^F, \quad T^{sl(2,R)/so(1,1)} = T - T^{so(1,1)}. \end{aligned}$$

Since $T^{so(1,1)}$ acts only on θ_L^F , the weight with respect to $T^{so(1,1)}$ is given by $\Delta^{so(1,1)}(J'_L) \equiv -J'^2_L/k + (\text{the grade of } J_{-n}^2 \text{'s})$. Moreover, for the untwisted sector, L_0 is given by the Casimir plus the total grade;

$$\Delta^{sl(2,R)/so(1,1)}(j, J_L) + \Delta^{so(1,1)}(J_L) = -\frac{j(j+1)}{k-2} + N,$$

where $-j(j+1)$ is the Casimir and N is the total grade of J_{-n}^a 's. Therefore, we find a general expression of L_0 ;

$$\begin{aligned} L_0 &= \Delta^{sl(2,R)/so(1,1)}(j, J_L) + \Delta^{so(1,1)}(J'_L) \\ &= \frac{-j(j+1)}{k-2} + \frac{J_L^2 - J'^2_L}{k} + N, \end{aligned}$$

where $\Delta^{sl(2,R)/so(1,1)}$ is the weight with respect to $T^{sl(2,R)/so(1,1)}$. Similarly, we obtain

$$\tilde{L}_0 = \frac{-j(j+1)}{k-2} + \frac{J_R^2 - J'^2_R}{k} + \tilde{N}.$$

Using these expressions, the level matching condition is given by

$$L_0 - \tilde{L}_0 = -n_w \left[(\Delta_- J_L + \Delta_+ J_R) - \frac{k}{2} n_w J_{BH} \right] + N - \tilde{N} \in \mathbf{Z}. \quad (4.17)$$

To proceed, let us consider the OPE of two vertex operators with quantum numbers $(n_{w,i}, J_{L,i}, J_{R,i})$ ($i = 1, 2$). Since $J_{L,R}$ and n_w are conserved, the level matching condition for the resulting operator reads

$$-(n_{w,1} + n_{w,2}) \sum_{i=1}^2 \left[(\Delta_- J_{L,i} + \Delta_+ J_{R,i}) - \frac{k}{2} n_{w,i} J_{BH} \right] \in \mathbf{Z}.$$

Thus, if $J_{L(R),1(2)}$ and $n_{W,1(2)}$ satisfy (4.17), the closure of the OPE requires [41]

$$(\Delta_- J_L + \Delta_+ J_R) - \frac{k}{2} n_w J_{BH} \equiv m_J \in \mathbf{Z}. \quad (4.18)$$

This is the solution to the level matching condition. The spectrum of the theory is specified by this condition.

We can check the single-valuedness of the vertex operator which satisfies this condition. Let us denote by $\exp(-i\Theta)$ the $\theta_{L,R}^{F,NF}$ -dependence of (4.15) and recall (4.12). Then, under $\delta\varphi = 2\pi$,

$$\delta\Theta = 2\pi m_J + \frac{k}{2}\pi n_w \left[\frac{1}{\pi} (\Delta_- \delta\theta_L^F - \Delta_+ \delta\theta_R^F) + (\Delta_+^2 - \Delta_-^2) \right].$$

Hence, the vertex operator is invariant under

$$\delta\theta_L^{NF} = \delta\theta_L^F = \pi\Delta_-, \quad \delta\theta_R^{NF} = \delta\theta_R^F = \pi\Delta_+.$$

Single-valuedness is guaranteed in this sense.

In our twisting, only the free field part seems relevant. In the untwisted sector, only the combinations $\theta_{L,R} = \theta_{L,R}^F + \theta_{L,R}^{NF}$ appear, so this does not matter. On the other hand, for a twisted sector, this is curious; we were originally considering the orbifolding with respect to $\varphi \sim \varphi + 2\pi$ including the non-free part. However, the non-free part *is* relevant in the above sense. This is related to the Noether current ambiguity in field theory [60]. In any case, one can take the point of view that we are just considering possible degrees of freedom represented by the twisting with respect to $\theta_{L,R}^F$.

So far we have dealt with a generic value of Δ_\pm corresponding to a rotating black hole. For the non-rotating black hole, we have only to set $\Delta_+ = \Delta_- = r_+$ in the above discussion. In addition, we can formally take the limit $\Delta_- \rightarrow 0$ at the end. However, we have to examine whether this limit in our result correctly represents the extremal limit as discussed before.

4.2.4 Physical states

Let us turn to the discussion on physical states [41]. We use the old covariant approach. The states corresponding to the vertex operators in (4.16) are written as

$$K_{-1}^a K_{-1}^b \cdots \tilde{K}_{-1}^c \tilde{K}_{-1}^d \cdots |j; J_L, n_w\rangle |j; J_R, n_w\rangle. \quad (4.19)$$

Here we have used the fact that the operators with higher grade are generated by those with grade 1. Then the physical states are given by the physical-state conditions

$$(L_n - \delta_n) |\Psi\rangle = (\tilde{L}_n - \delta_n) |\Psi\rangle = 0 \quad (n \geq 0).$$

In particular, the on-shell condition yields

$$\begin{aligned} J_L &= -\frac{k}{4}\Delta_- n_w + \frac{1}{\Delta_- n_w} \left(N - 1 - \frac{j(j+1)}{k-2} \right), \\ J_R &= +\frac{k}{4}\Delta_+ n_w - \frac{1}{\Delta_+ n_w} \left(\tilde{N} - 1 - \frac{j(j+1)}{k-2} \right), \\ N &= \tilde{N} + n_w m_J \end{aligned} \quad (4.20)$$

for twisted sectors ($n_w \neq 0$), and

$$1 = -\frac{j(j+1)}{k-2} + N, \quad N = \tilde{N} \quad (4.21)$$

for the untwisted sector ($n_w = 0$). Therefore, for a given j , an arbitrarily excited state is allowed in the twisted sectors. On the contrary, in the untwisted sector, j -value is completely determined by grade N :

$$j = j(N) \equiv \frac{1}{2} \left\{ -1 - \sqrt{1 + 4(k-2)(N-1)} \right\}, \quad (4.22)$$

where we have chosen the branch $\text{Re } j \leq -1/2$ (see appendix C). This result is the same as in the string theory on $SL(2, R)$.

4.3 Investigation of unitarity

We have discussed the spectrum of the $\widetilde{SL}(2, R)/Z_\varphi$ model by solving the level matching condition. But other consistency conditions remain to be discussed, and as a result, the spectrum in the previous section may be further restricted.

In this section, we will investigate the ghost problem. The unitary (ghost) problem for the string on $SL(2, R)$ has been discussed and it has been shown to contain ghosts [22],[28, 29]. However, there is a recent proposal for a unitary $SL(2, R)$ theory using modified currents [29]. Thus it may be worth studying our case. Because of the orbifolding and the use of representations in the hyperbolic basis, we cannot apply the argument in the $SL(2, R)$ theory to our case. Nevertheless, we can still utilize a tool developed for the $SL(2, R)$ theory with a slight modification. Here, we will first summarize the argument in the $SL(2, R)$ case. This may also make the later discussion clear. We then find explicit examples of negative-norm physical states; the string theory on $\widetilde{SL}(2, R)/Z_\varphi$ orbifold is not unitary [41].

4.3.1 The unitarity problem of a string on $SL(2, R)$

Let us briefly review the unitarity problem of the $SL(2, R)$ case [22],[28, 29]. The holomorphic and the anti-holomorphic part are independent in the $SL(2, R)$ WZW model until we consider the modular properties, so we focus on the holomorphic part. For the unitarity problem of the $SL(2, R)$ theory, it is useful to notice the following facts:

1. The on-shell condition is the same as (4.21).
2. Let V^a be an operator satisfying

$$[I_0^a, V^b] = i\epsilon^{ab}_c V^c,$$

(an example is $V^a = I_{-n}^a$) and consider the following states

$$V^+ I_0^- | j; m \rangle, \quad V^- I_0^+ | j; m \rangle, \quad V^0 | j; m \rangle.$$

Here $| j; m \rangle$ are eigenstates with the Casimir $\mathbf{C} = -j(j+1)$ and $I_0^0 = m$ (not necessarily base states). Moreover, assume they do not vanish. Then, by evaluating the matrix elements of the Casimir operator, one finds that these states are decomposed into the representations of $sl(2, R)$ with the j -values j and $j \pm 1$.

3. As a consequence of (2), acting I_{-1}^a N times on a base state $| j; m \rangle$ yields 3^N independent states at grade N with j -values ranging from $j - N$ to $j + N$. Let us call the states with $j \pm N$ the “extremal states” and denote them by $| E_N^\pm \rangle$. $| E_N^\pm \rangle$ are physical if they satisfy the on-shell condition. The reason is simple: Since the Casimir operator commutes with L_n , $L_n | E_N^\pm \rangle$ have the same j -value as $| E_N^\pm \rangle$. However, $L_n | E_N^\pm \rangle$ are at grade $N - n$, and thus their j -values should range from $j - (N - n)$ to $j + (N - n)$. Therefore, one has $L_n | E_N^\pm \rangle = 0$ ($n > 0$); together with the on-shell condition, they are physical.

4. Let $|\Psi\rangle$ be a physical state. Then the states obtained by acting J_0^a on $|\Psi\rangle$ are also physical:

$$(L_n - \delta_n) J_0^a \cdots J_0^b |\Psi\rangle = [(L_n - \delta_n), J_0^a \cdots J_0^b] |\Psi\rangle = 0 \quad (n \geq 0).$$

5. For the discrete series, one has a simple expression of the extremal states, e.g.,

$$| E_N^{d+} \rangle = (I_{-1}^+)^N | j(N); j(N) \rangle,$$

where $| j(N); j(N) \rangle$ is a highest-weight state, namely $I_0^+ | j(N); j(N) \rangle = 0$. Then it is easy to obtain the norms of these states:

$$\langle E_N^{d+} | E_N^{d+} \rangle = \langle j(N); j(N) | j(N); j(N) \rangle (N!) \prod_{r=0}^{N-1} (k + 2j(N) + r).$$

From (1)-(5), one immediately finds physical states with negative norms. First, let us consider the case $k < 2$. From (1) and (3), $| E_N^+ \rangle$ with $j = j(N)$ at its base is a physical state. At sufficiently large N , $j(N)$ takes a value of the principal continuous series. On the other hand, the j -value of $| E_N^+ \rangle$ is $j(N) + N$, but there is no unitary representations with this j -value. Thus, the module $I_0^a \cdots I_0^b | E_N^+ \rangle$ is physical, but forms a *non-unitary* representation of $sl(2, R)$.

Second, we consider the case $k > 2$. Again $| E_N^{d+} \rangle$ with $j = j(N)$ at its base is a physical state. In addition, one finds that

$$I_0^+ | E_N^{d+} \rangle = 0, \quad I_0^0 | E_N^{d+} \rangle = (j(N) + N) | E_N^{d+} \rangle.$$

Thus $|E_N^{d+}\rangle$ is a highest-weight state of a highest-weight $sl(2, R)$ representation like $|j(N) + N; j(N) + N\rangle$. However, the I^0 -value becomes positive for large N . Since there is no unitary representation of $sl(2, R)$ with such a highest weight state, the states in the module $I_0^a \cdots I_0^b |E_N^{d+}\rangle$ are physical but some have negative norms.

Although one can flip the sign of the norm of $|j(N); j(N)\rangle$ so that $\langle E_N^{d+} | E_N^{d+} \rangle > 0$ for arbitrary N , it is impossible to remove physical states with negative norms. This is because we have infinitely many physical states built on $|E_N^{d+}\rangle$ as in (4), and they form a non-unitary $sl(2, R)$ representation.

4.3.2 Physical states up to grade 1

Now we discuss the $\widetilde{SL}(2, R)/Z_\varphi$ orbifold case. One difference from the previous discussion is the existence of winding modes. Thus, for the twisted sectors, (4.22) does not hold and the holomorphic and anti-holomorphic part are not independent. Another important difference is that the Kac-Moody module is restricted to the form (4.19). We do not have states of the type in (4) and (5) in the previous subsection. Nevertheless, the discussion on the extremal states is still valid, so we will use them.

To proceed, let us consider physical states up to grade one. For the time being, we focus on the holomorphic part. At grade one, we have three states for a fixed j -, J_0^2 - and n_W - value;

$$|\pm\rangle \equiv K_{-1}^\pm |j; \lambda, n_W\rangle, \quad |2\rangle \equiv K_{-1}^2 |j; \lambda, n_W\rangle.$$

From the argument in appendix B, these states are decomposed into the eigenstates of the Casimir operator with j -values j and $j \pm 1$. We denote them by $|\Phi^j(j; \lambda, n_W)\rangle$ and $|\Phi^{j\pm 1}(j; \lambda, n_W)\rangle$. Note that $|\Phi^{j\pm 1}\rangle$ are the extremal states. Explicitly, they are given by (up to normalization) [41]

$$\begin{pmatrix} |\Phi^{j+1}\rangle \\ |\Phi^j\rangle \\ |\Phi^{j-1}\rangle \end{pmatrix} = \begin{pmatrix} j+1-i\lambda & -(j+1+i\lambda) & 2i((j+1)^2 + \lambda^2) \\ 1 & 1 & -2\lambda \\ -(j+i\lambda) & j-i\lambda & 2i(j^2 + \lambda^2) \end{pmatrix} \begin{pmatrix} |+\rangle \\ |-\rangle \\ |2\rangle \end{pmatrix}.$$

At grade one, the conditions $L_n = 0$ ($n > 0$) are reduced to $L_1 = 0$. This imposes one condition on a state given by a linear combination of $|\pm\rangle$ and $|2\rangle$. Then the space of the solution has (complex) two dimensions at a generic value of j and λ . Since we have the two extremal states satisfying $L_1 = 0$, the solutions take the form

$$\alpha |\Phi^{j+1}\rangle + \beta |\Phi^{j-1}\rangle. \quad (4.23)$$

At special values of λ and j , we have extra solutions. Similarly, we can get the states satisfying $\tilde{L}_1 = 0$ at grade one. Hence, from the states of the type (4.23) and base states, we obtain the physical states up to grade one by tensoring the holomorphic and the anti-holomorphic sector so that they satisfy the on-shell condition (4.20) or (4.21).

4.3.3 Non-unitarity of a string on $\widetilde{SL}(2, R)/Z_\varphi$ orbifold

Using the above physical states, we readily find physical states with negative norms [41]. First, let us discuss the case of real j (the complementary and the discrete series). In this case, we have the following physical states,

$$|\Psi_1^d\rangle = |j; J_{L,1}, 1\rangle |j; J_{R,1}, 1\rangle, \quad |\Psi_2^d\rangle = |\Phi^{j+1}(j; J_{L,2}, 1)\rangle |j; J_{R,2}, 1\rangle,$$

where $m_{J,1} = 0$, $m_{J,2} = 1$ and

$$\begin{aligned} J_{L,1} &= -\frac{k}{4}\Delta_- - \frac{1}{\Delta_-} \left(1 + \frac{j(j+1)}{k-2}\right), \quad J_{R,1} = \frac{k}{4}\Delta_+ + \frac{1}{\Delta_+} \left(1 + \frac{j(j+1)}{k-2}\right), \\ J_{L,2} &= -\frac{k}{4}\Delta_- - \frac{1}{\Delta_-} \frac{j(j+1)}{k-2}, \quad J_{R,2} = J_{R,1}. \end{aligned} \quad (4.24)$$

Taking into account the Hermiticity (4.8) and the action of $J_0^a(\tilde{J}_0^a)$, i.e., (C.6), we get the norms of these states by explicit calculation:

$$\begin{aligned} \langle \Psi_1^d | \Psi_1^d \rangle &= \langle j; J_{L,1}, 1 | j; J_{L,1}, 1 \rangle \langle j; J_{R,1}, 1 | j; J_{R,1}, 1 \rangle, \\ \langle \Psi_2^d | \Psi_2^d \rangle &= 2(j+1)(2j+1)(2j+k) \left((j+1)^2 + J_{L,2}^2 \right) \\ &\quad \times \langle j; J_{L,2}, 1 | j; J_{L,2}, 1 \rangle \langle j; J_{R,2}, 1 | j; J_{R,2}, 1 \rangle. \end{aligned}$$

$\langle j; J_{L,i}, 1 | j; J_{L,i}, 1 \rangle \langle j; J_{R,i}, 1 | j; J_{R,i}, 1 \rangle$ ($i = 1, 2$) take the same value if the bases of $|\Psi_1^d\rangle$ and $|\Psi_2^d\rangle$ belong to the same representation of $sl(2, R)$. Thus, for a sufficiently large $|j|$ (recall $j \leq -1/2$), the latter norm behaves as $8j^7/(k'\Delta_-)^2$, and the two norms have opposite signs. Although the j -value for the complementary series is restricted to $-1 < j \leq -1/2$, the discrete series appears by tensor products (see appendix C). In addition, bases with large $|j|$ are generated from those with small values by tensor products unless they decouple. Thus, if we include the bases with real j , our orbifold model cannot be unitary.

Next, we turn to the case of complex j (the principle continuous series). Because $j = -1/2 + i\nu$ ($\nu > 0$), the extremal states at grade one have $j = -1/2 \pm 1 + i\nu$. These correspond to complex Casimir values and non-unitary $sl(2, R)$ representations. This is not the end of the story however because (i) infinite series of states build on these states by the current zero-modes are not allowed and (ii) the left and right sector are connected by the quantum numbers n_w and m_J . Since the norm of $|\Psi_2^d\rangle$ vanishes in this case, we consider the following physical states instead:

$$\begin{aligned} |\Psi_1^p\rangle &= |j; J_L^1, 1\rangle |j; J_R^1, 1\rangle, \\ |\Psi_2^p\rangle &= \left(|\Phi^{j-1}(j; J_{L,2}, 1)\rangle - i |\Phi^{j+1}(j; J_{L,2}, 1)\rangle \right) |j; J_{R,2}, 1\rangle, \end{aligned}$$

where $J_{L(R),i}$ are given by (4.24). Again by explicit calculation, we get the norms of these states:

$$\begin{aligned}\langle \Psi_1^p | \Psi_1^p \rangle &= \langle j; J_{L,1}, 1 | j; J_{L,1}, 1 \rangle \langle j; J_{R,1}, 1 | j; J_{R,1}, 1 \rangle, \\ \langle \Psi_2^p | \Psi_2^p \rangle &= -4\nu \left[\left(J_{L,2}^2 - 1/4 - \nu^2 \right) \left(4\nu^2 - 3k - 1 \right) + 2(1+k)J_{L,2}^2 - k \right] \\ &\quad \times \langle j; J_{L,2}, 1 | j; J_{L,2}, 1 \rangle \langle j; J_{R,2}, 1 | j; J_{R,2}, 1 \rangle.\end{aligned}$$

Then, for a sufficiently large ν , the latter norm behaves as $-16\nu^7/(k'\Delta_-)^2$. Thus, the two norms have opposite signs if the bases of $|\Psi_1^p\rangle$ and $|\Psi_2^p\rangle$ belong to the same representation of $sl(2, R)$. Since bases with large ν are generated from those with small values by tensor products, our orbifold model is again non-unitary if we include the bases with complex j .

For the $SL(2, R)$ theory, a physical state at a sufficiently high grade has large $|j|$ at the base and it caused the trouble. In our case, some ghosts in the $SL(2, R)$ theory disappear, but physical states with large $|j|$ at the base exist already at grade one owing to the winding modes. This is because the winding modes can produce negative Virasoro weight. The existence of the ghost means that our model is not physical as it is. However, we have still possibilities that the orbifold model becomes ghost-free, for instance, by some truncation of the spectrum. We will discuss this issue in section 4.5.

4.4 Tachyon and target-space geometry

Before the consideration of the possibilities for a sensible theory, we will discuss general properties of the tachyon propagation and the target-space geometry which are irrelevant to the details of the full spectrum [41]. We have worked in an abstract framework based on representation theory so far, but we will find correspondences to the field theoretical approach in chapter 3. We see group theoretical meaning behind the black hole physics. In addition, we find properties similar to those in the $SL(2, R)/U(1)$ black hole theory because both theories are based on the $SL(2, R)$ WZW model and closely related.

4.4.1 Tachyon in the untwisted sector

First, we consider the tachyon in the untwisted sector. It is expressed by the matrix elements of $\widetilde{SL}(2, R)$ in unitary representations as (4.14). The matrix elements satisfy the differential equation [62]

$$[\Delta - j(j+1)] D_{J_L, J_R}^{j(\chi)}(g) = 0, \quad (4.25)$$

where Δ is the Laplace operator on $SL(2, R)$. Because the geometry of the black hole is locally $SL(2, R)$, this equation is nothing but the linearized tachyon equation in the black

hole geometry or the Klein-Gordon equation (3.4) up to a factor. For the untwisted sector, the on-shell condition is $-j(j+1)/(k-2) = 1$. Thus, at the critical value $k = 52/23$, the j -value corresponds to the principal continuous series, and the modes of the tachyon are given by (4.14) with $j = -1/2 + i/\sqrt{92}$.

Let us make an explicit correspondence between the tachyon and the scalar in chapter 3. Comparing (4.14) with (3.5), we find the correspondences

$$\begin{aligned} f(x) &\leftrightarrow {}^P D_{J_L \pm, J_R \pm}^\chi(g), & T_{\hat{E} \hat{N}} &\leftrightarrow {}^P D_{J_L \pm, J_R \pm}^\chi(g'), \\ -\hat{N} &\leftrightarrow J_L + J_R, & \hat{E} &\leftrightarrow J_L - J_R, \end{aligned}$$

where we have used (4.10). Since φ has period 2π , $N = r_- \hat{E} + r_+ \hat{N} \in \mathbf{Z}$. This confirms the level matching condition (4.18) with $n_w = 0$.

As a further check, let us consider the matrix elements for $g' = \begin{pmatrix} \cosh \rho/2 & \sinh \rho/2 \\ \sinh \rho/2 & \cosh \rho/2 \end{pmatrix}$ ($\rho > 0$); this corresponds to the region $r > r_+$. They are given by

$$\begin{aligned} {}^P D_{J_L+, J_R+}^\chi(g') &= \frac{1}{2\pi} B(\mu_L, -\mu_L - 2j) \frac{\cosh^{2j+\mu_L+\mu_R} \rho/2}{\sinh^{\mu_L+\mu_R} \rho/2} F(\mu_L, \mu_R; -2j; -\sinh^{-2} \rho/2), \\ {}^P D_{J_L-, J_R-}^\chi(g') &= \frac{1}{2\pi} B(1 - \mu_R, \mu_R - 1 + 2(j+1)) \frac{\cosh^{2j+\mu_L+\mu_R} \rho/2}{\sinh^{4j+2+\mu_L+\mu_R} \rho/2} \\ &\quad \times F(\mu_L + 2j + 1, \mu_R + 2j + 1; 2j + 2; -\sinh^{-2} \rho/2), \end{aligned} \quad (4.26)$$

where $\mu_{L,R} = iJ_{L,R} - j$. F and B are the hypergeometric function and the Euler beta function, respectively. Then from $-\sinh^2 \rho/2 = 1 - \hat{r}^2 = u$, we find that these are nothing but the mode functions in (3.7), i.e., $U_{\hat{E} \hat{N}}$ and $V_{\hat{E} \hat{N}}$ up to a phase. The “mass squared” μ and the j -value are related by $\mu = 4j(j+1)$.

Generically, the untwisted tachyon behaves as

$$\begin{aligned} {}^P D_{J_L+, J_R+}^\chi(g') &\sim a_1(r^2)^j && \text{as } r \rightarrow \infty, \\ &\sim b_1 e^{i(J_L - J_R) \ln \sqrt{r^2 - r_+^2}} + b_2 e^{-i(J_L - J_R) \ln \sqrt{r^2 - r_+^2}} && \text{as } r \rightarrow r_+, \\ {}^P D_{J_L-, J_R-}^\chi(g') &\sim a'_1(r^2)^{-(j+1)} && \text{as } r \rightarrow \infty, \\ &\sim b'_1 e^{i(J_L - J_R) \ln \sqrt{r^2 - r_+^2}} + b'_2 e^{-i(J_L - J_R) \ln \sqrt{r^2 - r_+^2}} && \text{as } r \rightarrow r_+, \end{aligned} \quad (4.27)$$

where $a_1^{(')}$ and $b_{1,2}^{(')}$ are certain constants. Since $\text{Re } j = -1/2$, they behave like spherical waves asymptotically. When $J_L = J_R$, the hypergeometric function degenerates and the asymptotic behaviors as $r \rightarrow r_+$ are different from (4.27).

4.4.2 Tachyon in the twisted sectors

Now we turn to the tachyon in the twisted sectors. The twisted tachyon is given by the product of the matrix elements and the twisting operator as (4.15). The twisting operator gives a phase to the tachyon. In the twisted sectors, various j -values are allowed from

the on-shell condition (4.20) with $m_J = N = \tilde{N} = 0$ and $n_w \neq 0$. Thus the matrix elements of the complementary and the discrete series appear as well as those of the principal continuous series. For the principal continuous and the complementary series, the explicit forms and the asymptotic behaviors of the matrix elements are given by the same expression as in (4.27) (although j -values are different).

For the discrete series, only one linear combination of the solutions to (4.25) appears. As explained in appendix C, the matrix elements are obtained from one of the matrix elements in the principal continuous series;

$${}^L D_{J_L, J_R}^j(g) \propto {}^H D_{J_L, J_R}^j(g) \propto {}^P D_{J_L+, J_R+}^\chi(g). \quad (4.28)$$

Thus we can read off the behaviors of ${}^{L,H} D_{J_L, J_R}^j(g')$ from ${}^P D_{J_L+, J_R+}^\chi(g')$. Note in particular that ${}^{L,H} D_{J_L, J_R}^j(g') \rightarrow (r^2)^j$ as $r \rightarrow \infty$ and $j \leq -1/2$. Therefore, a tachyon state in the discrete series damps rapidly as one goes to infinity, so this is a state localized near the black hole. This is similar to a winding state in the Euclidean $SL(2, R)/U(1)$ black hole where one can regard it as a bound state in the dual geometry [16]. Consequently, we have three kinds of the tachyon: One is from the principal continuous series and propagates like a wave, and another is from the complementary series and asymptotically behaves like r^{2j} or $r^{-2(j+1)}$ ($-1 < j \leq -1/2$), and the other is from the discrete series and is localized near the black hole.

For the untwisted tachyon, the tachyon scattering and the Hawking radiation have been discussed in [52, 64]. These arguments are also valid for the tachyon from the principal series in our case. In addition, the tachyon modes from the discrete series are the same as $U_{\hat{E}\hat{N}}$ with $\mu \geq -1$. Thus most of the discussion in chapter 3 is valid for the tachyon from the discrete series. Finally, the tachyon states satisfy the condition at infinity (3.8) except for a part of the tachyon from the complementary series.

4.4.3 Global properties

So far we have not discussed the global properties of the tachyon, but considered the tachyon propagation in one patch of the orbifold (the region $r > r_+$). In order to discuss the tachyon propagation globally, we have to continue it from one region to another. Let us start with a tachyon in region I ($r > r_+$). Then the tachyon is given by a linear combination of (4.26) or (4.28) and is regular at infinity. From the linear transformation formulas of the hypergeometric function, we can obtain the expression around $r = r_\pm$ as in (4.27). We would like to continue it to the other regions.

Here we have two possible sources of obstacles. One is the complex power of u or $1-u$. The other is the logarithmic singularities like $\ln u$ or $\ln(1-u)$. These cause troubles as $u \rightarrow 0$ ($r \rightarrow r_+$) or $u \rightarrow 1$ ($r \rightarrow r_-$). The logarithmic singularity at $u = 0$ arises when

$\mu_L - \mu_R \in \mathbf{Z}$, i.e., $J_L - J_R = 0$, and the one at $u = 1$ arises when $\mu_L + \mu_R + 2j \in \mathbf{Z}$, i.e., $J_L + J_R = 0$. The latter corresponds to the case of the $SL(2, R)/U(1)$ black hole in which the tachyon develops a logarithmic singularity at the origin (singularity) [16]. This is natural because the inner horizon of the $\widetilde{SL}(2, R)/Z_\varphi$ black hole and the origin of the $SL(2, R)/U(1)$ black hole are the same point in the $SL(2, R)$ group manifold.

Note that the matrix elements are continuous over the entire group manifold. Thus if we consider a generalized function space including distributions, we can continue the tachyon from one region to another in any case. Similarly, we can discuss the global properties of the scalar fields in chapter 3 by using representation theory of $\widetilde{SL}(2, R)$. These considerations about tachyon fields may be useful for further investigations of the low energy theory.

4.4.4 T-duality

Finally, we will briefly discuss the properties under T-duality transformations. The $\widetilde{SL}(2, R)/Z_\varphi$ black hole has two Killing vectors $\partial_{\hat{t}}$ and $\partial_{\hat{\varphi}}$. In the coordinate system $(\hat{t}, \hat{\varphi}, \hat{r})$, the geometry is given by (4.1) and the dilaton $\phi = 0$. In order to deal with a general T-duality transformation, let us define new coordinates x and y by

$$\begin{pmatrix} \hat{t} \\ \hat{\varphi} \end{pmatrix} = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}, \quad \alpha\delta - \beta\gamma \neq 0.$$

Then, the T-duality transformation with respect to ∂_x covers all the T-duality transformations.

First, let us consider the T-duality transformation with respect to ∂_φ . This has been discussed in [33]. Setting $x = \varphi$ and $y = t$, the duals of the $\widetilde{SL}(2, R)/Z_\varphi$ black holes become in general black strings. Thus this T-duality transformation is not self-dual.

Next, let us set $x = \hat{\varphi}$ and $y = \hat{t} - \hat{\varphi}$. In these coordinates, the geometry is given by

$$\begin{aligned} ds_{string}^2 &= \alpha' k \left\{ dx^2 + (1 - \hat{r}^2) dy^2 + 2(1 - \hat{r}^2) dx dy + (\hat{r}^2 - 1)^{-1} d\hat{r}^2 \right\}, \\ B &= \alpha' k \hat{r}^2 dx \wedge dy, \quad \phi = 0. \end{aligned}$$

Then from the formula of T-duality transformations [65, 66], we get the following dual geometry [41]:

$$\begin{aligned} \tilde{ds}_{string}^2 &= \alpha' k \left\{ dx^2 + \hat{r}^2 dy^2 + 2\hat{r}^2 dx dy + (\hat{r}^2 - 1)^{-1} d\hat{r}^2 \right\}, \\ \tilde{B} &= \alpha' k (1 - \hat{r}^2) dx \wedge dy, \quad \tilde{\phi} = 0. \end{aligned}$$

This geometry is obtained from the original one also via $\hat{r}^2 \rightarrow 1 - \hat{r}^2$ or $\hat{t} \leftrightarrow \hat{\varphi}$. Thus, this T-duality transformation is self-dual and interchanges the inside of the outer horizon

$(\hat{r}^2 < 1)$ and the outside of the inner horizon (or the outside of the origin for the non-rotating black hole) $(\hat{r}^2 > 0)$. In particular, the outer and the inner horizon (or the origin) are interchanged. Recall that translations of \hat{t} and $\hat{\varphi}$ are the vector and the axial symmetry. So, the transformation $\hat{t} \leftrightarrow \hat{\varphi}$ corresponds to the T-duality transformation in the $SL(2, R)/U(1)$ black hole which interchanges these symmetries and also the horizon and the singularity [16, 14].

Since φ is periodic, we have to further specify the periodicity of the dual coordinate. In the above T-duality transformation, the period of $x = \hat{\varphi}$ in the dual geometry should be reciprocal of that in the original geometry [66]. From (2.4), we see that the periods of \hat{t} and $\hat{\varphi}$ are not independent, so generically, we cannot specify the period of $\hat{\varphi}$ only. However, for the non-rotating black hole ($r_- = 0$), we have $\hat{\varphi} = r_+ \varphi$ and the period of $\hat{\varphi}$ in the original geometry is equal to $2\pi r_+$. Hence the period in the dual geometry is $2\pi/(r_+ k)$. This indicates that the black hole mass is reversed under the T-duality transformation because $M_{BH} = r_+^2$. Since $J_{L,R}$ take all real values, the spectrum of L_0 and \tilde{L}_0 is formally invariant under this T-duality transformation. But it is not bounded from below as in Minkowski spacetime, so we need some procedure such as the Wick rotation for a rigorous argument.

4.5 Discussion

In this chapter, we developed the string theory in the three dimensional black hole geometry in the framework of conformal field theory. This was the first attempt to quantize a string in a black hole background with an infinite number of propagating modes. The model was described by an orbifold of the $\widetilde{SL}(2, R)$ WZW model. We constructed the orbifold. We discussed the spectrum by solving the level matching condition and obtained winding modes. We also analyzed the physical states and examined the ghost problem. We found explicit examples of negative-norm physical states. We then discussed the tachyon propagation and the target-space geometry. We found correspondence between the group theoretical approach and the field theoretical one in the previous chapter. We also found a self-dual T-duality transformation reversing the black hole mass. Although problems still remain, our results may serve as a starting point for further investigations.

The existence of the negative-norm physical states indicates that our model is not physical as it stands. Therefore, in the following, we will consider possibilities for obtaining a sensible theory after a brief discussion on consistency conditions other than the ghost problem.

4.5.1 Consistency conditions

The basic physical consistency conditions for a string theory are not many. In general, as a sensible physical theory, we must require Lorentz invariance, a positive inner product for the observable Hilbert space and the unitary transition amplitude. There are only a few in number, but these in turn imply various consistency conditions such as world-sheet diffeomorphism and Weyl invariance, the absence of negative-norm states, closure of OPE, level matching and modular invariance. Even though the absence of a tachyon might also be added to the list, the presence of a tachyon in the bosonic string does not indicate any fundamental inconsistency in the theory.¹³ In addition, for modular invariance, it is sufficient to check associativity of OPE and modular invariance of the one-point amplitude at one-loop [68].

It does not seem easy for a string theory to satisfy all these requirements. However, there is a common belief that a world-sheet anomaly (either local or global) always leads to a spacetime anomaly.¹⁴ So, a string theory is likely to be automatically consistent once world-sheet anomalies are removed.

With these general remarks in mind, we comment on several consistency conditions in our case [41].

Closure of OPE

Unitarity requires the closure of OPE, and the fusion rules are determined by tensor products of the underlying primaries and by non-trivial null states in the Kac-Moody and the Virasoro module. We need detailed studies of these modules in order to find the condition from the non-trivial null states. But it is easy to find that from the tensor products. The tensor products of the unitary representations are summarized in appendix C. From them, we find that the tensor products are closed if the content of the operators is given by (i) only the highest (or the lowest) discrete series, (ii) the highest, lowest discrete series and the principal continuous series, or (iii) all the unitary series, so that addition and subtraction of the j -values are closed mod \mathbf{Z} .

Partition function and modular invariance

Next, we turn to modular invariance. From the spectrum in section 4.2, we get

$$\begin{aligned} L_0 - \tilde{L}_0 &= -n_w m_J + N - \tilde{N}, \\ L_0 + \tilde{L}_0 &= \frac{-2j(j+1)}{k-2} + N + \tilde{N} - n_w \left(\frac{k}{2} \Delta_+^2 n_w - 2\Delta_+ J_R + m_J \right). \end{aligned}$$

¹³See however Ref. [67], which might imply that the bosonic string does not exist nonperturbatively.

¹⁴Some works on this theme are as follows: the connection of the modular invariance and spacetime anomalies are discussed in [69] (for the type I) and [70] (for the type II and the heterotic string); the connection between the modular invariance and unitarity are discussed in [69, 71].

Then the partition function diverges since the Casimir $-j(j+1)$, J_R and two integers n_w, m_J can take arbitrarily large or small values. In Minkowski spacetime, we can avoid the divergence of the partition function by the Wick rotation, but we have no analog in our case. Furthermore, our Kac-Moody module is restricted to the states of the form (4.19), so we have to take this into account in the character calculation.

One resolution to this problem might be to find a subclass of the spectrum and/or to develop an analog of the Wick rotation so that we get a finite and modular invariant partition function. This might also solve the ghost problem. For compact group manifolds [61], the spectrum is restricted to integrable representations of the Kac-Moody algebra, so that one can get modular invariant partition functions. Fields in non-integrable representations decouple in correlators. However, the argument depends largely upon compactness, so we have to take different strategies for non-compact cases. So far, there is no general argument, but, for the $SL(2, R)$ theory, there are a few attempts[25]-[27]. Besides group manifolds, a partition function of a string theory in a curved spacetime is discussed in [30].

4.5.2 *Toward a sensible theory*

Finally, let us discuss possibilities for obtaining a sensible string theory in the three dimensional black hole background [41]. We can speculate various reasons why ghosts survive in our analysis:

1. Further truncation might be necessary on the spectrum.
2. Modular invariance might fix the problem.
3. The theory based on $SL(2, R)$ might be sick. The $SL(2, R)$ WZW model describes anti-de Sitter space, so has unusual asymptotic properties.
4. One might have to use modified currents.
5. We might have to include non-unitary representations for base representations of current algebras.

All of the possibilities listed above appear in the discussion on the $SL(2, R)$ and the $SL(2, R)/U(1)$ theory [24, 25],[29],[17]. However, the possibility (5) does not work: even if we include non-unitary representations, our argument in section 4.3 does not change very much and we can easily find physical states with negative norms. From general remarks in the previous subsection, the most plausible solution to our ghost problem is the possibility (2). This might be related to (1). However, the modular invariance for a string theory in a curved spacetime is a hard problem as we saw in the above. Here, we will discuss the possibility (1) which is different from previously discussed ones, and (4).

Discrete symmetries

One possibility to consistently truncate the spectrum is further orbifolding besides that with respect to $\varphi \sim \varphi + 2\pi$. As we will see, only a part of the $\widetilde{SL}(2, R)$ manifold is necessary for describing the three dimensional black hole. Since we have started from the $\widetilde{SL}(2, R)$ WZW model, the redundant part of the manifold should be divided away by orbifolding. Let us discuss the relevant discrete symmetries [41].

In appendix C, we see that the $SL(2, R)$ manifold contains sixteen domains denoted by $\pm D_i^\pm$ ($i = 1-4$). One correspondence between Region I-III and these domains is

$$\text{Region I} = D_1^+, \quad \text{Region II} = D_2^- \cup (-D_3^+), \quad \text{Region III} = -D_4^-.$$

Here we have taken a parametrization in Region II and III slightly different from the one in section 4.1, but the geometry is the same. Thus we need only the universal covering space of the region $\Omega_1 \equiv D_1^+ \cup D_2^- \cup (-D_3^+) \cup (-D_4^-)$ to get the black hole geometry as long as we do not consider its maximal extension. Now let us define two transformations by

$$\begin{aligned} T_1 : \quad g &\rightarrow \quad g' = -g, \\ T_2 : \quad g &\rightarrow \quad g' = \mathcal{B}g \quad \text{in } \pm D_{1,2}^\pm, \quad g' = -\mathcal{B}g \quad \text{in } \pm D_{3,4}^\pm, \end{aligned}$$

where \mathcal{B} is given by (C.13) and called Bargmann's automorphism of $SL(2, R)$. $T_{1,2}$ have the properties

$$\begin{aligned} T_1^2 = T_2^2 &= 1, \\ T_1 : \quad \Omega_{1(2)} &\rightarrow -\Omega_{1(2)}, \quad T_2 : \quad \Omega_{1(2)} \rightarrow \Omega_{2(1)}, \end{aligned}$$

where $\Omega_2 = (D_1^- \cup D_2^+ \cup D_3^- \cup D_4^+)$. Note that $\pm \Omega_{1,2}$ cover all the sixteen domains of $SL(2, R)$ and have no overlap among them. Moreover we can obtain the black hole geometry from each of the four sets as in section 4.1. Thus we can divide $SL(2, R)$ by the \mathbf{Z}_2 symmetries, T_1 and T_2 , in order to drop redundant regions.

There is one more discrete symmetry. This is related to the problem of closed timelike curves. Region I-III or each of $\pm \Omega_{1,2}$ includes the region $r^2 < 0$ where closed timelike curves exist [31]. This region corresponds to part of $-D_4^-$ in Ω_1 for the rotating case or the whole region for the non-rotating case. Although we have no symmetry to remove this region only, it is possible to drop it together with the region $(r_+^2 + r_-^2)/2 > r^2 > 0$. The region $(r_+^2 + r_-^2)/2 > r^2$ corresponds to $(-D_3^+) \cup (-D_4^-)$ in Ω_1 , so we have only to find a symmetry between $D_1^+ \cup D_2^+$ and $(-D_3^+) \cup (-D_4^-)$. The symmetry is easy to find in the coordinate system $(\hat{t}, \hat{\varphi}, \hat{r})$. Let us define a \mathbf{Z}_2 transformation by

$$T_3 : \quad (\hat{t}, \hat{\varphi}, \hat{r}^2 - 1/2) \rightarrow (\hat{\varphi}, \hat{t}, -(\hat{r}^2 - 1/2)).$$

Then the geometry given by (4.1) is invariant under T_3 . This symmetry maps any point in $D_1^+ \cup D_2^+$ ($\hat{r}^2 > 1/2$) to a point in $(-D_3^+) \cup (-D_4^-)$ ($\hat{r}^2 < 1/2$) and vice versa. Thus we can truncate both the spectrum and the region with closed timelike curves by the orbifolding with respect to T_3 at the expense of the additional dropped region. Notice that a part of T_3 , i.e., $\hat{r}^2 \rightarrow 1 - \hat{r}^2$ or $\hat{t} \leftrightarrow \hat{\varphi}$, has already appeared in the discussion of the T-duality in section 4.4.

The use of modified currents

Now we turn to another possibility. In the flat theory, the no-ghost theorem has been proved [72]. Thus it seems useful to consider the flat limit of our model and observe how the ghosts disappear. However, we cannot take this limit: The three dimensional flat theory is described by three free bosons. Hence, e.g. for the left sector, there are three pairs of conjugate zero-modes, and the base states are specified by three momenta as $|p^0, p^1, p^2\rangle$. On the other hand, the base states of our model, e.g. in the left sector, have only two labels as $|j; J\rangle$ (although the total labels for both the left and the right sector are three). Because of the deficiency of the zero-modes, we cannot get to the flat theory.

The deficiency of the zero-modes is observed from a different point of view. Recall the Wakimoto realization of the $sl(2, R)$ Kac-Moody algebra [73]. It is realized by a free boson ϕ and a β - γ ghost system¹⁵:

$$\begin{aligned} iJ^+(z) &= \beta(z), \\ iJ^-(z) &= \gamma^2\beta(z) + \sqrt{2k'}\gamma\partial\phi(z) + k\partial\gamma(z), \\ iJ^2(z) &= \gamma\beta(z) + \sqrt{k'/2}\partial\phi(z), \end{aligned}$$

where $k' \equiv k - 2$ and

$$\begin{aligned} \beta(z)\gamma(w) &= -\gamma(z)\beta(w) \sim \frac{1}{z-w}, \\ \phi(z)\phi(w) &\sim -\ln(z-w). \end{aligned}$$

The β - γ ghosts can be bosonized by two free bosons [74], but some of the zero-modes of these bosons are redundant. The redundant zero-modes are related to the picture changing of the ghost system and absent from the original algebra.

On the other hand, there is an argument based on effective action that a string in a nearly flat AdS_3 ($SL(2, R)$) with weak curvature must be unitary [75]. Therefore, it may be possible to construct a unitary $SL(2, R)$ and $\widetilde{SL}(2, R)/Z_\varphi$ theory if we incorporate the deficient zero-modes so that the model has the flat limit. Indeed, we may re-interpret

¹⁵This representation is slightly different from the one diagonalizing J^0 .

Bars' argument for ghost-free spectrum of a $SL(2, R)$ theory [29] along this line of thought. He realizes the β - γ ghosts by two free fields

$$\beta = \partial\phi^+, \quad \gamma = \phi^-,$$

where $\phi^\pm = (1/\sqrt{2})(\phi^0 \pm \phi^1)$ and $\phi^i(z)\phi^j(w) \sim (-1)^i \delta^{ij} \ln(z-w)$ ($i = 0, 1$). Owing to the redundant zero-modes, the currents are modified. However, by a careful treatment of the zero-modes, one can show that the current algebra is maintained, the string on $SL(2, R)$ has no ghosts, and the flat theory is recovered in the limit $k \rightarrow \infty$.

We cannot apply his realization to the string theories on $SL(2, R)/Z_\varphi$ or $SL(2, R)/U(1)$: For the $SL(2, R)$ WZW model, the allowed states in his realization are only certain combinations of the left and the right sector which diagonalize J_0^+ (\tilde{J}_0^+). For the black hole cases, we need the states diagonalizing J_0^2 (\tilde{J}_0^2). However, it is interesting to generalize his argument and apply it to the black hole physics [76].

5 CONCLUSION

In this thesis, we discussed quantum aspects of the three dimensional black holes. In chapter 3, we considered scalar fields with a generic mass squared in the three dimensional black hole background, and discussed their thermodynamics in the framework of quantum field theory in curved spacetime. We took two approaches. One was based on mode expansion and summation over states. In the other approach, we used Hartle-Hawking Green functions. We obtained exact expressions of mode functions, the Hartle-Hawking Green functions, Green functions on a cone geometry, and thermodynamic quantities. These constitute a reliable basis of the quantum field theory and the thermodynamics of scalar fields in the three dimensional black hole background. Our results did not necessarily agree with those in the literature and the thermodynamic quantities depended largely upon their definitions, boundary conditions and regularization schemes. These indicate the importance of curvature effects and precise discussions. We may need further investigations of this issue in particular for the cases of finite black hole mass (i.e., truly curved cases). Our model may be useful for this purpose.

In chapter 4, we considered the string theory in the three dimensional black hole geometry in the framework of conformal field theory. This was the first attempt to quantize a string theory in a black hole background with an infinite number of propagating modes. We constructed an orbifold of the $\widetilde{SL}(2, R)$ WZW model, which described the string in the three dimensional black hole geometry. We discussed the spectrum by solving the level matching condition and obtained winding modes. We also analyzed the physical

states and found negative-norm physical states. The tachyon and the target-space geometry were discussed. The existence of the negative-norm physical states implies that our model is not sensible as it stands. Thus we discussed possibilities to obtain a sensible string theory. Our detailed analyses may serve as a basis for further investigations of this subject.

We still have difficulties both in the thermodynamics and in the string theory. Because of them, the analyses in chapter 3 and 4 are not fully connected yet. However, we believe that our results may provide useful insights into quantum aspects of the three dimensional quantum black holes.

We are now about to directly catch gravitational waves. A large amount of data concerning cosmology is accumulating. Moreover, we have seen an interesting result in super string theory that black holes work well as a probe into quantum gravity [46, 77]. I sincerely hope that, together with these developments, further investigations of quantum black holes lead to a deeper understanding of quantum theory of gravity.

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APPENDIX

A The Feynman Green function in \widetilde{AdS}_3

In appendix A, we summarize the derivation of the Feynman Green function in the universal covering space of three dimensional anti-de Sitter space (\widetilde{AdS}_3). Quantization of a scalar field in \widetilde{AdS}_D has been discussed in [49]-[51], and the Feynman Green function has been obtained [49, 51] in terms of the hypergeometric function. In the three dimensional case, the Feynman Green function is simplified and expressed in terms of elementary functions [35].

\widetilde{AdS}_3 is defined by its embedding in a four dimensional flat space of signature $(- - + +)$ through the equation

$$-x_0^2 - x_1^2 + x_2^2 + x_3^2 = -l^{-2}.$$

We parametrize this by

$$\begin{aligned} x_0 &= l \sin \tau \sec \rho, & x_1 &= l \cos \tau \sec \rho, \\ x_2 &= l \sin \theta \tan \rho, & x_3 &= l \cos \theta \tan \rho, \end{aligned}$$

where $0 \leq \rho < \pi/2$, $0 \leq \theta < 2\pi$, $-\infty < \tau < \infty$. Then the metric becomes

$$ds^2 = l^2 \sec^2 \rho \left(-d\tau^2 + d\rho^2 + \sin^2 \rho \, d\theta^2 \right).$$

The field equation for a scalar field is given by

$$(\square - \mu l^{-2}) \psi(x) = 0,$$

where $\square = \frac{1}{\sqrt{-g}} \partial_\mu \sqrt{-g} g^{\mu\nu} \partial_\nu$. Making the separation of variables

$$\psi(x) = \sum \psi_{m\omega} = \sum e^{-i\omega\tau} e^{im\theta} R_{m\omega}(\rho), \quad (m \in \mathbf{Z}),$$

the equation for the radial function $R_{m\omega}(\rho)$ is written as

$$\left(\partial_\rho^2 + \frac{1}{\sin \rho \cos \rho} \partial_\rho + \omega^2 - \frac{m^2}{\sin^2 \rho} - \mu \sec^2 \rho \right) R_{m\omega}(\rho) = 0.$$

We make a further change of variables $v = \sin^2 \rho$, and define a function $f_{m\omega}(v)$ by

$$R_{m\omega}(\rho) = v^{|m|/2} (1-v)^{\lambda/2} f_{m\omega}(v),$$

with

$$\lambda = \lambda_\pm \equiv 1 \pm \sqrt{1 + \mu}.$$

Then the radial equation is reduced to the hypergeometric equation

$$[v(1-v)\partial_v^2 + \{c - (a+b+1)v\}\partial_v - ab] f_{m\omega}(v) = 0,$$

where

$$\begin{aligned} a &= \frac{1}{2}(\lambda + |m| - \omega), \\ b &= \frac{1}{2}(\lambda + |m| + \omega), \\ c &= |m| + 1. \end{aligned}$$

(We will deal with the case of real λ , i.e., $\mu \geq -1$.) If we require the regularity at $v = 0$, the solution is expressed by the Gauss' hypergeometric function F as

$$f_{m\omega}(v) = F(a, b; c; v).$$

Since \widetilde{AdS}_3 is not globally hyperbolic, it is necessary to impose boundary conditions at spatial infinity. Following [49, 50], we impose the condition to conserve energy. This means that the surface integral of the energy-momentum tensor at spatial infinity must vanish. This requirement leads to

$$|\omega| = \lambda + |m| + 2n \quad (n = 0, 1, 2, \dots),$$

where

$$\lambda = \begin{cases} \lambda_{\pm} & \text{for } 0 > \mu > -1, \\ \lambda_{+} & \text{for } \mu \geq 0, \mu = -1 \end{cases}.$$

Then the value of a takes zero or a negative integer. By using a mathematical formula [78], one obtains

$$\begin{aligned} \psi(x) &= \sum_{m,n} [a_{mn}\psi_{mn} + (a_{mn}\psi_{mn})^*] \quad (m \in \mathbf{Z}, \quad n = 0, 1, 2, \dots), \\ \psi_{mn} &= C_{mn} e^{-i\omega\tau} e^{im\theta} (\sin \rho)^{|m|} (\cos \rho)^{\lambda} P_n^{(|m|, \lambda-1)}(\cos 2\rho), \end{aligned} \quad (\text{A.1})$$

where $P_n^{(\alpha, \beta)}$ are Jacobi Polynomials and C_{mn} are normalization constants.

For the positive frequency part $\psi^{(+)}$ of the solution one can define a positive definite scalar product by

$$(\psi_1^{(+)}, \psi_2^{(+)}) \equiv -i \int_{\Sigma} d^2x \sqrt{-g} g^{0\nu} \psi_1^{(+)*} \overleftrightarrow{\partial}_{\nu} \psi_2^{(+)},$$

where Σ is a spacelike surface. Then the normalization constant C_{mn} is determined by the condition $(\psi_{mn}^{(+)}, \psi_{m'n'}^{(+)}) = \delta_{mm'}\delta_{nn'}$. By using the orthogonal relation with respect to the Jacobi Polynomials [78],

$$\begin{aligned} &\int_0^{\pi/2} d\rho \tan \rho (\sin \rho)^{2|m|} (\cos \rho)^{2\lambda} P_n^{(|m|, \lambda-1)}(\cos 2\rho) P_{n'}^{(|m|, \lambda-1)}(\cos 2\rho) \\ &= \delta_{nn'} \frac{1}{2(2n + \lambda + |m|)} \frac{\Gamma(n + |m| + 1)\Gamma(n + \lambda)}{n!\Gamma(n + \lambda + |m|)}, \end{aligned} \quad (\text{A.2})$$

one obtains

$$C_{mn} = \left[\frac{n! \Gamma(|m| + \lambda + n)}{2\pi l(|m| + n)! \Gamma(\lambda + n)} \right]^{1/2}.$$

Now we quantize the scalar field by setting the commutation relation

$$[a_{mn}, a_{m'n'}^\dagger] = \delta_{mm'} \delta_{nn'}.$$

Then one finds

$$\begin{aligned} [\psi(x), \psi(x')]_{\tau=\tau'} &= 0, \\ [\psi(x), \partial_{\tau'} \psi(x')]_{\tau=\tau'} &= -i \frac{1}{g^{\tau\tau} \sqrt{-g}} \delta(\theta - \theta') \delta(\rho - \rho'). \end{aligned} \quad (\text{A.3})$$

Here we have used the orthogonal relation (A.2). The δ function is defined for the space of functions of the form (A.1).

We then define the Feynman Green function by

$$\begin{aligned} -iG_F(x, x') &= \langle 0 | T \{ \psi(x) \psi(x') \} | 0 \rangle \\ &\equiv \theta(\tau - \tau') \sum_{m,n} \psi_{mn}(x) \psi_{mn}^*(x') + (x \leftrightarrow x'). \end{aligned}$$

From (A.3), one can check

$$(\square - \mu l^{-2}) G_F(x, x') = \frac{1}{\sqrt{-g}} \delta(x - x'). \quad (\text{A.4})$$

Furthermore, one can perform the summation with respect to m and n . First, we set $x' = (\tau', \rho', \theta') = (0, 0, 0)$ (i.e., $(x'_0, x'_1, x'_2, x'_3) = (0, l, 0, 0)$) without loss of generality because \widetilde{AdS}_3 is homogeneous. Then only the terms with $m = 0$ contribute to the summation;

$$-iG_F(x, 0) = \frac{1}{2\pi l} e^{-i\lambda|\tau|} (\cos \rho)^\lambda \sum_{n=0}^{\infty} e^{-2in|\tau|} P_n^{(0, \lambda-1)}(\cos 2\rho).$$

By making use of the mathematical formula [79]

$$\begin{aligned} \sum_{k=0}^{\infty} \frac{(\alpha + \beta + 1)_k}{(\beta + 1)_k} t^k P_k^{(\alpha, \beta)}(x) \\ = (1+t)^{-\alpha-\beta-1} F \left(\frac{\alpha + \beta + 1}{2}, \frac{\alpha + \beta + 2}{2}; \beta + 1; \frac{2t(x+1)}{(t+1)^2} \right), \end{aligned}$$

one obtains [51]

$$-iG_F(x, 0) \equiv -iG_F(z) = \frac{l^{-1}}{2^{\lambda+1} \pi} z^{-\lambda} F \left(\frac{1}{2} \lambda, \frac{1}{2} (\lambda + 1); \lambda; z^{-2} \right).$$

Here z is defined by

$$z = 1 + l^{-2} \sigma(x, 0) + i\varepsilon. \quad (\text{A.5})$$

$\sigma(x, x')$ is half of the distance between x and x' in the four dimensional flat space,

$$\sigma(x, x') = \frac{1}{2} \eta_{\alpha\beta} (x - x')^\alpha (x - x')^\beta,$$

where $\eta_{\alpha\beta}$ and x^α ($\alpha, \beta = 0-3$) are the metric and the coordinates of the flat space, respectively. The infinitesimal imaginary part $i\varepsilon$ ($\varepsilon > 0$) in z was added so that the Green function looked locally like the Minkowski one [49]. In the three dimensional case, by the formula,

$$F\left(a, \frac{1}{2} + a; 2a; z\right) = 2^{2a-1} (1-z)^{-1/2} \left[1 + (1-z)^{1/2}\right]^{1-2a},$$

the Feynman Green function is simplified to [35]

$$-iG_F(z) = \frac{l^{-1}}{4\pi} (z^2 - 1)^{-1/2} \left[z + (z^2 - 1)^{1/2}\right]^{1-\lambda}.$$

This result is obtained also by replacing $|\tau|$ with $|\tau| - i\varepsilon$ so that $|e^{-2in|\tau|}| < 1$ and by utilizing the generating function of the Jacobi Polynomials.

For a generic x' , we have only to replace $\sigma(x, 0)$ with $\sigma(x, x')$.

B The Sommerfeld representation of Green functions

In appendix B, we derive $G_F^E(x, x'_n; \beta)$ in chapter 3 and its derivative with respect to β [35].

B.1 Derivation of $G_F^E(x, x'_n; \beta)$

We begin with the definition

$$\tilde{G}_F^E(\zeta; 2\pi) \equiv G_F(z(\zeta, i\Delta\varphi_n^{E+}; r, r'); \beta_H) \Big|_{\Delta\varphi_n^{E+}, r, r': \text{fixed}}.$$

By definition, $\tilde{G}_F^E(w_n; 2\pi) = G_F^E(x, x'_n; \beta_H)$ where w_n is given by (3.14). $\tilde{G}_F^E(\zeta; 2\pi)$ depends upon ζ through

$$\begin{aligned} z(\zeta, i\Delta\varphi_n^{E+}; r, r') - i\varepsilon &= \frac{1}{d_H^2} \left[\sqrt{r^2 - r_-^2} \sqrt{r'^2 - r_-^2} \cosh\left(\frac{ir_+}{l} \Delta\varphi_n^{E+}\right) \right. \\ &\quad \left. - \sqrt{r^2 - r_+^2} \sqrt{r'^2 - r_+^2} \cosh(i\zeta) \right]. \end{aligned} \quad (\text{B.1})$$

Thus $\tilde{G}_F^E(\zeta; 2\pi)$ is periodic under $\zeta \rightarrow \zeta + 2\pi$ ($\tau \rightarrow \tau + \beta_H$). $z = \pm 1$ are the points of its singularities. On ζ -plane, there are four points corresponding to these singularities

in the region $-\pi < \operatorname{Re} \zeta \leq \pi$. They are indicated by crosses (\times) in Figure 2. These points are symmetric with respect to the origin $\zeta = 0$, and infinitesimally close to the imaginary axis for $\Delta\varphi_0^{E+} = 0$, i.e., when we take the trace of Green functions. The authors of [55] discussed how to construct a Green function with an arbitrary period for certain differential equations. By following them, we get the Sommerfeld integral representation of the Green functions with an arbitrary period in our case [35]:

$$\tilde{G}_F^E(w_n; 2\pi\beta/\beta_H) = \frac{\beta_H}{2\pi\beta} \int_{\Gamma} d\zeta \tilde{G}_F^E(\zeta; 2\pi) \frac{e^{i\beta_H\zeta/\beta}}{e^{i\beta_H\zeta/\beta} - e^{i\beta_H w_n/\beta}}, \quad (\text{B.2})$$

where the contour Γ is given by the solid lines in Figure 2. This contour consists of two parts and divides the four singularities into two pairs. Then by recovering other variables, we obtain

$$G_F^E(x, x'_n; \beta) = \tilde{G}_F^E(w_n; 2\pi\beta/\beta_H).$$

It is instructive to consider some special cases before we show the validity of the above expression. First, we consider the case of $\beta = \beta_H/q$, ($q = 1, 2, \dots$). Notice that the contour Γ can be deformed into Γ' given by the dashed lines in Figure 2. Since the integrand is of period 2π in this case, the contributions from the path made up of straight lines cancel with each other. Thus only the residues inside the circular path contribute to the integral. Then we get

$$\tilde{G}_F^E(w_n; 2\pi/q) = \sum_k \tilde{G}_F^E(w_n(k); 2\pi), \quad (\text{B.3})$$

where $w_n(k)$ and $k(\in \mathbf{Z})$ are given by $w_n(k) = w_n + 2\pi k/q$ and $-\pi < w(k) \leq \pi$. In this case, the method of images works and we can explicitly check the periodicity. Clearly, the right-hand side of (B.3) reproduces $\tilde{G}_F^E(w_n; 2\pi)$ for $q = 1$.

Next, we consider the case $\beta \rightarrow \infty$. In the limit $\beta \rightarrow \infty$, the expression (B.2) is reduced to

$$\tilde{G}_F^E(w_n; \infty) = \frac{1}{2\pi i} \int_{\Gamma} \tilde{G}_F^E(\zeta; 2\pi) \frac{d\zeta}{\zeta - w_n}. \quad (\text{B.4})$$

From the formula $\lim_{n \rightarrow \infty} \sum_{k=-n}^n 1/(x+k) = \pi \cot \pi x$, we obtain another expression of $\tilde{G}_F^E(w_n; 2\pi\beta/\beta_H)$ [35]:

$$\begin{aligned} \tilde{G}_F^E(w_n; 2\pi\beta/\beta_H) &= \sum_{k=-\infty}^{\infty} \tilde{G}_F^E(w_n + 2\pi k\beta/\beta_H; \infty) \\ &= \frac{\beta_H}{4\pi i \beta} \int_{\Gamma} d\zeta \tilde{G}_F^E(\zeta; 2\pi) \cot \left\{ \frac{\beta_H}{2\beta} (\zeta - w_n) \right\}. \end{aligned} \quad (\text{B.5})$$

The equivalence to the former expression is easily checked by noting $\tilde{G}_F^E(w_n; \beta_H) = \tilde{G}_F^E(w_n; -\beta_H)$.

Now we check the properties necessary for the Green function with a generic β . First, $\tilde{G}_F^E(w_n; 2\pi\beta/\beta_H)$ ($G_F^E(x, x'_n; \beta)$) actually converges because $\tilde{G}_F^E(\zeta; 2\pi)$ comes to vanish exponentially as $|\text{Im } \zeta| \rightarrow \infty$. The periodicity of $\tilde{G}_F^E(w_n; 2\pi\beta/\beta_H)$ is easily confirmed by (B.5). Finally, let us check that $G_F^E(x, x'; \beta)$ satisfies the inhomogeneous equation. Remember (the Euclidean version of) (A.4), then from (B.4) we find that

$$(\square^E - \mu)G_F^E(x, x'; \infty) = \frac{a}{\sqrt{|g^E|}} \delta_\infty^E(x - x') ,$$

where $a = -1$ for $J_{BH} = 0$ and $a = i$ for $J_{BH} \neq 0$. Here we have explicitly denoted the period of the delta function with respect to τ . Then using $G_F^E(x, x'; \beta) = \sum_{k=-\infty}^{\infty} G_F^E(x, x'; \infty) \Big|_{\Delta\tau \rightarrow \Delta\tau + k\beta}$, we get the desired result

$$(\square^E - \mu l^{-2})G_F^E(x, x'; \beta) = \frac{a}{\sqrt{|g^E|}} \delta_\beta^E(x - x') .$$

B.2 Derivation of $\partial_\beta G_F^E(x, x'_n; \beta)|_{\beta=\beta_H}$

To calculate the entropy, we need $\partial_\beta G_F^E(x, x'_n; \beta)$. From the integral representation (B.5), we have

$$\begin{aligned} \frac{\partial}{\partial \beta} G_F^E(x, x'_n; \beta) &= -\frac{1}{\beta} G_F^E(x, x'_n; \beta) \\ &\quad + \frac{\beta_H^2}{8\pi i \beta^3} \int_{\Gamma} d\zeta \tilde{G}_F^E(\zeta; 2\pi)(\zeta - w_n) \text{cosec}^2 \left\{ \frac{\beta_H}{2\beta}(\zeta - w_n) \right\} . \end{aligned} \quad (\text{B.6})$$

For $\beta = \beta_H$, the above expression is fairly simplified. First, we deform the contour Γ into Γ' . Within the circular path, there is only one singularity at $\zeta = w_n$. The contribution from the residue of this singularity cancels with the first term in (B.6). Then by changing variables to $i\zeta' = \zeta \pm \pi$ according to the left and right straight path of Γ' , we get

$$\frac{\partial}{\partial \beta} G_F^E(x, x'_n; \beta) = \frac{1}{4\beta_H} \int_{-\infty}^{\infty} d\zeta' \frac{\tilde{G}_F^E(i\zeta' - \pi; 2\pi)}{\cos^2 \{(i\zeta' - w_n)/2\}} .$$

Note that $\tilde{G}_F^E(i\zeta' - \pi; \beta_H)$ is a function of

$$z(\zeta') \equiv z(i\zeta' - \pi, i\Delta\varphi_n^{E+}; r, r') = A_n + B \cosh \zeta' ,$$

where

$$A_n = \frac{1}{d_H^2} \sqrt{r^2 - r_-^2} \sqrt{r'^2 - r_-^2} \cosh \left(\frac{ir_+}{l} \Delta\varphi_n^{E+} \right) , \quad B = \frac{1}{d_H^2} \sqrt{r^2 - r_+^2} \sqrt{r'^2 - r_+^2} .$$

We then make the further change of variables from ζ' to z , and use

$$\frac{dz}{d\zeta'} = B \sinh \zeta' = \pm \sqrt{(z - A_n)^2 - B^2} ,$$

and

$$2 \cos^2 \left\{ \frac{1}{2} (i\zeta' - w_n) \right\} = c_n \frac{z - A_n}{B} \pm s_n \frac{\sqrt{(z - A_n)^2 - B^2}}{B} + 1,$$

where

$$c_n = \cosh(iw_n), \quad s_n = \sinh(iw_n).$$

Consequently, we get the fairly simple expression [35]

$$\begin{aligned} & \frac{\partial}{\partial \beta} G_F^E(x, x'_n; \beta) \Big|_{\beta=\beta_H} \\ &= -\frac{B}{\beta_H} \int_{A_n+B}^{\infty} dz \, G_F^E(z; \beta_H) \frac{1}{\sqrt{(z - A_n)^2 - B^2}} \frac{c_n(z - A_n) + B}{(z - A_n + c_n B)^2}. \end{aligned}$$

C Representations of $SL(2, R)$

In this appendix, we briefly summarize the representation theory of $SL(2, R)$ (and of its universal covering group $\widetilde{SL}(2, R)$) and collect its useful properties for discussions in this thesis. For a review, see [62] and [80]-[82].

C.1 $SL(2, R)$

Preliminary

The group $SL(2, R)$ is represented by real matrices

$$g = \begin{pmatrix} a & b \\ c & d \end{pmatrix}, \quad ad - bc = 1.$$

It has one-parameter subgroups

$$\Omega_a = \{g_a(t) = e^{-it\tau^a}\}, \quad a = 0, 1, 2,$$

where

$$\begin{aligned} \tau^0 &= -\frac{1}{2}\sigma_2 \rightarrow g_0(t) = \begin{pmatrix} \cos t/2 & \sin t/2 \\ -\sin t/2 & \cos t/2 \end{pmatrix}, \\ \tau^1 &= \frac{i}{2}\sigma_1 \rightarrow g_1(t) = \begin{pmatrix} \cosh t/2 & \sinh t/2 \\ \sinh t/2 & \cosh t/2 \end{pmatrix}, \\ \tau^2 &= \frac{i}{2}\sigma_3 \rightarrow g_2(t) = \begin{pmatrix} e^{t/2} & 0 \\ 0 & e^{-t/2} \end{pmatrix}, \end{aligned}$$

and σ_i ($i = 1-3$) are the Pauli matrices. In Ω_0 , $g_0(0)$ and $g_0(4\pi)$ represent the same point and $g_0(t)$, $t \in [0, 4\pi]$, traces an uncontractable loop in $SL(2, R)$. If one decompactifies

this loop and does not identify $g_0(0)$ and $g_0(4\pi)$, one obtains the universal covering group $\widetilde{SL}(2, R)$. $\tau^a (a = 0, 1, 2)$ have the properties

$$[\tau^a, \tau^b] = i\epsilon^{ab}_c \tau^c, \quad \text{Tr} (\tau^a \tau^b) = -\frac{1}{2} \eta^{ab},$$

where $\eta^{ab} = \text{diag} (-1, 1, 1)$. These form a basis of $sl(2, R)$.

$SL(2, R)$ is isomorphic to $SU(1, 1)$ (and so is $sl(2, R)$ to $su(1, 1)$). An isomorphism is given by

$$\tilde{g} = T^{-1} g T, \quad T = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & i \\ i & 1 \end{pmatrix},$$

where $\tilde{g} \in SU(1, 1)$ and $g \in SL(2, R)$. Note that \tilde{g}_0 is diagonal in $SU(1, 1)$, while so is g_2 in $SL(2, R)$.

Parametrization

Any matrix g of $SL(2, R)$, with all its elements being non-zero, can be represented as

$$g = d_1 (-e)^{\epsilon_1} s^{\epsilon_2} p d_2.$$

Here, $\epsilon_{1,2} = 0$ or 1 ; $d_i = \text{diag} (e^{\psi_i/2}, e^{-\psi_i/2})$ ($i = 1, 2$);

$$-e = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}, \quad s = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix};$$

and p is one of the following matrices

$$\begin{aligned} p &= g_1(\theta), \quad -\infty < \theta < +\infty, \\ p &= g_0(\theta), \quad -\pi/2 < \theta < +\pi/2. \end{aligned}$$

Thus, $SL(2, R)$ has eight domains given by

$$\begin{aligned} D_1 &= \left\{ A_1 = \begin{pmatrix} e^\phi \cosh \theta/2 & e^\psi \sinh \theta/2 \\ e^{-\psi} \sinh \theta/2 & e^{-\phi} \cosh \theta/2 \end{pmatrix}, \quad -\infty < \theta < +\infty \right\}, \\ D_2 &= \left\{ A_2 = \begin{pmatrix} e^\phi \cos \theta/2 & e^\psi \sin \theta/2 \\ -e^{-\psi} \sin \theta/2 & e^{-\phi} \cos \theta/2 \end{pmatrix}, \quad -\frac{\pi}{2} < \theta < +\frac{\pi}{2} \right\}, \\ D_3 &= \left\{ A_3 = \begin{pmatrix} -e^\phi \sin \theta/2 & e^\psi \cos \theta/2 \\ -e^{-\psi} \cos \theta/2 & -e^{-\phi} \sin \theta/2 \end{pmatrix}, \quad -\frac{\pi}{2} < \theta < +\frac{\pi}{2} \right\}, \\ D_4 &= \left\{ A_4 = \begin{pmatrix} e^\phi \sinh \theta/2 & e^\psi \cosh \theta/2 \\ -e^{-\psi} \cosh \theta/2 & -e^{-\phi} \sinh \theta/2 \end{pmatrix}, \quad -\infty < \theta < +\infty \right\}, \\ -D_i &= \{-A_i\} \quad (i = 1 \sim 4), \end{aligned}$$

where $-\infty < \phi, \psi < +\infty$. One can further divide these domains according to the sign of θ . We denote the domains with positive θ by $\pm D_i^+$ and those with negative θ by $\pm D_i^-$.

When a matrix element of g is zero, it is, for example, written by $\begin{pmatrix} a & 0 \\ b & a^{-1} \end{pmatrix}$. Taking appropriate limits of $\pm A_i$ yields such a matrix.

C.2 Unitary representations

Let us denote the generators of $sl(2, R)$ by J^a and consider the basis given by $I^0 = J^0$ and $I^\pm = J^1 \pm iJ^2$. In this basis, the non-trivial commutation relations read

$$[I^0, I^\pm] = \pm I^\pm, \quad [I^+, I^-] = -2I^0.$$

This basis is natural from the $su(1, 1)$ point of view because I^0 corresponds to diagonal elements and I^\pm are regarded as ladder operators as in $su(2)$. Using this basis, one can classify all unitary representations of $sl(2, R)$ and hence those of $SL(2, R)$ and $\widetilde{SL}(2, R)$ [80],[62, 23]. There are five classes of the unitary representations of $sl(2, R)$ which are labeled by the Casimir $\mathbf{C} = \eta_{ab}J^aJ^b$, I^0 and a parameter $m_0 \in [0, 1)$:

1. Principal continuous series T_χ^P : Representations realized in $\{ |j, m\rangle\}$, $m = m_0 + k$, $0 \leq m_0 < 1$, $k \in \mathbf{Z}$ and $j = -1/2 + i\nu$, $0 < \nu$.
2. Complementary (Supplementary) series T_χ^C : Representations realized in $\{ |j, m\rangle\}$, $m = m_0 + k$, $0 \leq m_0 < 1$, $k \in \mathbf{Z}$, and $\min\{-m_0, m_0 - 1\} < j \leq -1/2$.
3. Highest weight discrete series T_j^H : Representations realized in $\{ |j, m\rangle\}$, $m = M_{max} - k$, $k \in \mathbf{Z}_{\geq 0}$ and $j = M_{max} \leq -1/2$ such that $I^+ |j, j\rangle = 0$.
4. Lowest weight discrete series T_j^L : Representations realized in $\{ |j, m\rangle\}$, $m = M_{min} + k$, $k \in \mathbf{Z}_{\geq 0}$ and $j = -M_{min} \leq -1/2$ such that $I^- |j, -j\rangle = 0$.
5. Identity representation : The trivial representation $| -1, 0 \rangle$.

Here, χ is the pair (j, m_0) ; $\mathbf{Z}_{\geq 0}$ refers to non-negative integers; and we have denoted the value of \mathbf{C} by $-j(j+1)$. Note that j need not be real although $-j(j+1)$ should be and that one can restrict j to $\text{Im } j > 0$ for (1) and $j \leq -1/2$ for the others because j and $-(j+1)$ represent the same Casimir.

Unitary representations of $\widetilde{SL}(2, R)$ are realized in the same space $\{ |j, m\rangle\}$. For $SL(2, R)$, the parameters are further restricted to $m_0 = 0, 1/2$ in (1), $m_0 = 0$ in (2) and $j = (\text{half integers})$ in (3) and (4). We will use the same notations for the groups as in $sl(2, R)$.

From the harmonic analysis on $\widetilde{SL}(2, R)$, we find that a complete basis for the square integrable functions on $\widetilde{SL}(2, R)$ is given by the matrix elements of the principal continuous series, the highest and lowest weight discrete series.

C.3 Tensor products

Because one has various unitary representations, the decomposition of tensor products is more complicated than $SU(2)$. Basic strategy to get the decomposition is to decompose the tensored representation spaces into the eigenspaces of the Casimir operator [83, 84]. We are interested in tensor products among the unitary representations. Then, for $SL(2, R)$, the decompositions are given as follows [62, 85]:

1. For two discrete series of the same type,

$$T_{j_1}^{L,H} \otimes T_{j_2}^{L,H} = \sum_{n=0}^{\infty} \oplus T_{j_1+j_2-n}^{L,H}.$$

2. For two discrete series of different types,

$$T_{j_1}^L \otimes T_{j_2}^H = \int_0^{\infty} T_{(-1/2+i\rho, m_0)}^P d\mu(\rho) \oplus \sum_{j=-m_0-1}^{j_1-j_2} (T_j^L \oplus T_j^H),$$

where $m_0 = j_1 - j_2 \bmod \mathbf{Z}$ and $d\mu(\rho)$ is a continuous measure. We have assumed $j_2 \geq j_1$, but the opposite case is obtained similarly. We remark that $j \leq -m_0 - 1$ and the identity representation does not appear in the right-hand side [85].¹⁶

3. For a discrete and a principal continuous series,

$$T_{j_1}^{L,H} \otimes T_{(-1/2+i\rho', m'_0)}^P = \int_0^{\infty} T_{(-1/2+i\rho, m_0)}^P d\mu(\rho) \oplus \sum_{j=-m_0-1}^{-\infty} T_j^{L,H},$$

where $m_0 = m'_0 + j_1 \bmod \mathbf{Z}$.

4. For two principal continuous series,

$$\begin{aligned} & T_{(-1/2+i\rho', m'_0)}^P \otimes T_{(-1/2+i\rho'', m''_0)}^P \\ &= \int_0^{\infty} T_{(-1/2+i\rho, m_0)}^P d\mu_1(\rho) \oplus \int_0^{\infty} T_{(-1/2+i\rho, m_0)}^P d\mu_2(\rho) \oplus \sum_{j=-m_0-1}^{-\infty} (T_j^L \oplus T_j^H), \end{aligned} \tag{C.1}$$

where $m_0 = m'_0 + m''_0 \bmod \mathbf{Z}$.

The tensor product of a principal and a complementary series, or that of two complementary series is decomposed into principal and discrete series like (C.1) [83, 84]. In the latter, one complementary series appears additionally in certain cases. The tensor product of a complementary and a discrete series is similar to that of a principal and a discrete series [84].

¹⁶In [86], it is claimed that the identity representation does appear as an exceptional case. In our understanding, they show just the existence of the solution to the recursion equation for the Clebsch-Gordan coefficients.

The decompositions are determined essentially by local properties of the group as is clear from the consideration of tensor products of $sl(2, R)$. Thus the decompositions for $\widetilde{SL}(2, R)$ are obtained by continuing the value of m_0 and j .

The Clebsch-Gordan coefficients have been discussed in [86, 85], [62, 82], [18].

C.4 Representations in the hyperbolic basis

In appendix C.2, we have discussed the representations in the basis diagonalizing $J^0 = I^0$ which is the compact direction of $SL(2, R)$. One can also consider the basis diagonalizing J^2 or $J^- = J^0 - J^1$ which are the non-compact directions [62], [85], [87]-[90], [16]. The generators J^0 , J^2 and J^- are called elliptic, hyperbolic and parabolic, respectively. An outstanding feature of the non-compact generators is that they have continuous spectrum. In what follows, we will concentrate on representations in the hyperbolic basis.

In terms of $J^\pm \equiv J^0 \pm J^1$ and J^2 , the commutation relations are given by

$$[J^+, J^-] = -2iJ^2, \quad [J^2, J^\pm] = \pm iJ^\pm. \quad (\text{C.2})$$

The latter equation indicates that the ladder operators J^\pm change the eigenvalue of J^2 by $\pm i$. This seems to contradict the Hermiticity of J^2 . However, this is not the case [87]: In general, the eigenvalue of an Hermitian operator with *continuous* spectrum need not be real [91].

For our purpose, however, it is convenient to choose spectrum with real values. Thus, we use the basis of a representation space given by $\{|\lambda\rangle\}$, where λ is the eigenvalue of J^2 and runs through all the real number. For the principal continuous and the complementary series, the eigenvalue of J^2 has multiplicity two. So, the basis has an index \pm to distinguish them and is given by $\{|\lambda\rangle_\pm\}$. We will omit this and the other indices to specify representations such as j, m_0, L and H unless we need them. In the above basis, an element (a state) of the representation space is given by a “wave packet”

$$|\phi\rangle = \int_{-\infty}^{\infty} d\lambda \phi(\lambda) |\lambda\rangle, \quad \|\phi\|^2 = \int_{-\infty}^{\infty} d\lambda |\phi(\lambda)|^2 < \infty.$$

This is analogous to a state in field theory where one uses a plane wave basis in infinite space. Then the generators act on the state as

$$\begin{aligned} J^2 |\phi\rangle &= \int_{-\infty}^{\infty} d\lambda \lambda \phi(\lambda) |\lambda\rangle, \\ J^+ |\phi\rangle &= \int_{-\infty}^{\infty} d\lambda f_+(\lambda) \phi(\lambda - i) |\lambda\rangle, \\ J^- |\phi\rangle &= \int_{-\infty}^{\infty} d\lambda f_-(\lambda + i) \phi(\lambda + i) |\lambda\rangle. \end{aligned} \quad (\text{C.3})$$

f_\pm play the role of the matrix elements in this basis. From the above action, the commutation rules are realized if

$$f_+(\lambda) f_-(\lambda) - f_-(\lambda + i) f_+(\lambda + i) = -2i\lambda. \quad (\text{C.4})$$

An eigenstate $|\lambda'\rangle$ is obtained in the limit $\phi(\lambda) \rightarrow \delta(\lambda - \lambda')$.

It is possible to introduce $|\lambda \pm i\rangle$ and write the action of the generators as

$$\begin{aligned} J^+ |\phi\rangle &= \int_{-\infty}^{\infty} d\lambda f_+(\lambda + i)\phi(\lambda) |\lambda + i\rangle, \\ J^- |\phi\rangle &= \int_{-\infty}^{\infty} d\lambda f_-(\lambda)\phi(\lambda) |\lambda - i\rangle, \\ J^+ |\lambda\rangle &= f_+(\lambda + i) |\lambda + i\rangle, \quad J^- |\lambda\rangle = f_-(\lambda) |\lambda - i\rangle. \end{aligned}$$

In this way, one can formally consider eigenstates $|\lambda \pm i\rangle$. However, we should always understand them in the sense of (C.3). Note that $|\lambda \pm i\rangle$ can be “expanded” by the original basis $\{|\lambda\rangle\}$, where $\lambda \in \mathbf{R}$.

Now let us consider the matrix elements of J^\pm . In the elliptic basis, the Casimir operator takes the form

$$\begin{aligned} \mathbf{C} &= \eta_{ab} J^a J^b \\ &= -I^0(I^0 + 1) + I^- I^+ = -I^0(I^0 - 1) + I^+ I^-, \end{aligned}$$

and the actions of $I^\pm I^\mp$ are given by

$$I^- I^+ |j; m\rangle = \tilde{d}^2(j, m) |j; m\rangle, \quad I^+ I^- |j; m\rangle = \tilde{d}^2(j, m - 1) |j; m\rangle,$$

where $\tilde{d}^2(j, m) = -j(j + 1) + m(m + 1)$. Then one obtains the norms of $I^\pm |j; m\rangle$ and hence the matrix elements of I^\pm . In the hyperbolic basis, the final step dose not work because $(J^\pm)^\dagger = J^\pm$. In this case, the Casimir operator takes the form

$$\mathbf{C} = J^2(J^2 + i) - J^- J^+ = J^2(J^2 - i) - J^+ J^-, \quad (\text{C.5})$$

and the actions of $J^\pm J^\mp$ are given by

$$J^+ J^- |j; \lambda\rangle = d^2(j, \lambda - i) |j; \lambda\rangle, \quad J^- J^+ |j; \lambda\rangle = d^2(j, \lambda) |j; \lambda\rangle, \quad (\text{C.6})$$

where

$$d^2(j, \lambda) \equiv \lambda(\lambda - i) + j(j + 1).$$

Note that $d^2(j, \lambda - i) = \overline{d^2(j, \lambda)}$ and these actions satisfy (C.4). One cannot determine the matrix elements of J^\pm (i.e., f_\pm) separately without additional conditions. We see that (C.2), (C.5) and (C.6) are related to the corresponding equations in the elliptic basis by the “analytic continuation” $J^\pm \rightarrow -iI^\pm$ and $J^2 \rightarrow iI^0$ [87].

C.5 Matrix elements

By explicit realization of the representations in spaces of functions, one can calculate the matrix elements of $SL(2, R)$. Here we consider the matrix elements in the hyperbolic basis [62], [89, 90], [16].

First, let us discuss the principal continuous series T_χ^P of $SL(2, R)$. This representation is realized in a space of functions on a real axis, \mathcal{I}_χ . The action of the group element $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2, R)$ and the inner product are given by

$$(T_\chi^P(g)f)(x) = |bx + d|^{2j} \operatorname{sign}^{2m_0}(bx + d) f\left(\frac{ax + c}{bx + d}\right), \quad (\text{C.7})$$

$$(f_1(x), f_2(x)) = \int_{-\infty}^{\infty} dx \overline{f_1(x)} f_2(x). \quad (\text{C.8})$$

Then one finds that

$$\psi_{\lambda\pm}^\chi(x) \equiv \frac{1}{\sqrt{2\pi}} x^{-i\lambda+j} \theta(\pm x), \quad \lambda \in \mathbf{R}, \quad (\text{C.9})$$

form an orthonormal basis diagonalizing the action of J^2 , namely,

$$\begin{aligned} (\psi_{\lambda\epsilon}^\chi(x), \psi_{\mu\epsilon'}^\chi(x)) &= \delta_{\epsilon\epsilon'} \delta(\lambda - \mu), \\ [T_\chi^P(g_2(t)) \psi_{\lambda\pm}^\chi](x) &= e^{-it\lambda} \psi_{\lambda\pm}^\chi(x), \quad g_2(t) \in \Omega_2, \end{aligned}$$

where $\epsilon, \epsilon' = \pm$. $\psi_{\lambda\pm}^\chi$ correspond to $|\lambda\rangle_\pm$ in the previous subsection and are not elements in \mathcal{I}_χ .

One can calculate the matrix elements in the basis (C.9) using (C.7) and (C.8). For example, for $t > 0$ one has

$$\begin{aligned} {}^P D_{\lambda+, \lambda'+}^\chi(g_1(t)) &= \frac{1}{2\pi} B(\mu, -\mu' - 2j) \frac{\cosh^{2j+\mu+\mu'} t/2}{\sinh^{\mu+\mu'} t/2} \\ &\quad \times F(\mu, \mu'; -2j; -\sinh^{-2} t/2), \end{aligned} \quad (\text{C.10})$$

$$\begin{aligned} {}^P D_{\lambda-, \lambda'-}^\chi(g_1(t)) &= \frac{1}{2\pi} B(1 - \mu', \mu' - 1 + 2(j + 1)) \frac{\cosh^{2j+\mu+\mu'} t/2}{\sinh^{4j+2+\mu+\mu'} t/2} \\ &\quad \times F(\mu + 2j + 1, \mu' + 2j + 1; 2j + 2; -\sinh^{-2} t/2), \end{aligned} \quad (\text{C.11})$$

$${}^P D_{\lambda\epsilon, \lambda'\epsilon'}^\chi(g_2(t)) = e^{-it\lambda} \delta_{\epsilon\epsilon'} \delta(\lambda - \lambda'), \quad (\text{C.12})$$

where $\mu^{(\prime)} = i\lambda^{(\prime)} - j$. F and B are the hypergeometric and the Euler beta function, respectively. For $g_1(t)$, ${}^P D_{\lambda-, \lambda'+}^\chi$ is given by a linear combination of (C.10) and (C.11), and ${}^P D_{\lambda+, \lambda'-}^\chi$ vanishes.

The matrix elements for the complementary series are obtained by analytically continuing the value of j [89].

Let us turn to the discrete series T_j^L . This is realized in a space of analytic functions on \mathbf{C}_+ (the upper half-plane). (This can also be embedded in the principal continuous series.) The action of $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2, R)$ and the inner product are given by ¹⁷

$$\begin{aligned} (T_j^L(g)f)(w) &= (bw + d)^{2j} f\left(\frac{aw + c}{bw + d}\right), \\ (f_1(w), f_2(w)) &= \frac{i}{2\Gamma(-2j-1)} \int_{\mathbf{C}_+} dw d\bar{w} y^{-2j-2} \overline{f_1(w)} f_2(w), \end{aligned}$$

where $w = x + iy$ and $dwd\bar{w} = -2idxdy$. One then finds that

$$\varphi_\lambda^j(w) = \frac{1}{2^{(j+1)}\pi} e^{-\lambda\pi/2} \Gamma(-i\lambda - j) w^{-i\lambda+j}, \quad \lambda \in \mathbf{R},$$

form an orthonormal basis diagonalizing J^2 . Thus similarly to the previous case (or using the fact that $f(w)$ is determined by its values on the semi-axis $w = iy$ ($y > 0$)), one obtains the matrix elements. ${}^L D_{\lambda,\lambda'}^j(g_1(t))$ is the same up to a numerical factor as (C.10) and ${}^L D_{\lambda,\lambda'}^j(g_2(t))$ is given by (C.12) without $\delta_{\epsilon\epsilon'}$.

For the highest weight series T_j^H , one can get the matrix elements from those of the lowest weight series. By utilizing an automorphism of $SL(2, R)$ called Bargmann's automorphism of $SL(2, R)$

$$\mathcal{B} : \begin{pmatrix} a & b \\ c & d \end{pmatrix} \rightarrow \begin{pmatrix} a & -b \\ -c & d \end{pmatrix}, \quad (\text{C.13})$$

the matrix elements of the highest weight series are given by [62, 89]

$${}^H D_{\lambda,\lambda'}^j(g) = {}^L D_{\lambda,\lambda'}^j(\mathcal{B}g).$$

All the matrix elements satisfy the differential equation

$$[\Delta - j(j+1)] D_{\lambda,\lambda'}^{j(\chi)}(g) = 0,$$

where Δ is the Laplace operator on $SL(2, R)$ and they are characterized essentially by local properties of $SL(2, R)$. Hence, the matrix elements of $\widetilde{SL}(2, R)$ are obtained by continuing the values of j and m_0 .

D Decomposition of the Kac-Moody module

The Clebsch-Gordan decomposition similar to $su(2)$ holds for $sl(2, R)$ ($su(1, 1)$) in the elliptic basis [22]. This argument is valid for the hyperbolic basis as well with a slight modification. In appendix D, we will show this [41].

¹⁷ $j = -1/2$ case needs special treatment, but the matrix elements take the same forms as in $j < -1/2$ cases [80, 89].

Let V^a be a vector operator, i.e.,

$$[J_0^a, V^b] = i\epsilon^{ab}{}_c V^c,$$

and $| j; \lambda \rangle$ be an eigenstate of \mathbf{C} and J^2 . An example is $V^a = J_{-n}^a$. $| j; \lambda \rangle$ need not be a base state of the Kac-Moody module. Then let us consider states

$$V^+ J_0^- | j; \lambda \rangle, \quad V^- J_0^+ | j; \lambda \rangle, \quad V^2 | j; \lambda \rangle. \quad (\text{D.1})$$

In the hyperbolic basis, these states do not vanish in any unitary representation. From (C.5), the matrix elements of the Casimir operator with respect to these states are

$$\mathbf{C} = \begin{pmatrix} c + 2i\lambda & 0 & i \\ 0 & c - 2i\lambda & -i \\ -2id^2(j, \lambda - i) & 2id^2(j, \lambda) & c - 2 \end{pmatrix}, \quad \text{where } c = -j(j + 1).$$

The trace and determinant in this subspace are given by

$$\text{Tr } \mathbf{C} = 3c - 2, \quad \det \mathbf{C} = c^2(c + 2).$$

In addition, it is easy to see that the state $(1, 1, -2\lambda)$ is an eigenvector with the Casimir $\mathbf{C} = -j(j + 1)$. Thus, the other eigenvalues are $-j(j - 1)$ and $-(j + 1)(j + 2)$ and the states in (D.1) are decomposed into the $sl(2, R)$ representations with j -values j and $j \pm 1$. The corresponding eigenvectors ψ_j and $\psi_{j \pm 1}$ are given by

$$\begin{aligned} \psi_j &= (1, 1, -2\lambda), \\ \psi_{j-1} &= (-(j + i\lambda), j - i\lambda, 2i(j^2 + \lambda^2)), \\ \psi_{j+1} &= (j + 1 - i\lambda, -(j + 1 + i\lambda), 2i((j + 1)^2 + \lambda^2)). \end{aligned}$$

Note that ψ_{j+1} is obtained from ψ_{j-1} by the replacement $j \rightarrow -j - 1$.

Here, it may be useful to remark on the norm of states [22]. Consider representations where the Casimir operator is Hermitian. The representations need not be unitary. Furthermore, let $|\Psi_1\rangle$ and $|\Psi_2\rangle$ be eigenstates with the Casimir values c_1 and c_2 , respectively. Then by evaluating the matrix element $\langle \Psi_1, \mathbf{C} \Psi_2 \rangle = (\mathbf{C} \Psi_1, \Psi_2)$, one obtains

$$(\bar{c}_1 - c_2) \langle \Psi_1 | \Psi_2 \rangle = 0.$$

Therefore, for complex c_1 and c_2 , the norm vanishes when $c_1 = c_2$. It can be non-zero only when c_1 and c_2 are complex conjugate. Since the extremal states built on a principal continuous series have complex Casimir values (see section 4.3), they become physical states with zero norm. On the other hand, $\langle E_N^+ | E_N^- \rangle$ can be non-zero because their Casimir values are complex conjugate. (Thus these extremal states are not null.)

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